MATH 242 Engineering Mathematics II

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Chapter 1

Differential Equations

1.1 Introduction

This Section can be found in the Section 2.3 of the Textbook

1.1.1 Newton's Law of Cooling

A: Ambient Temperature (assume constant)
T: Fluid Temperature (depends on time t)

$$\frac{dT}{dt} = k(A - T) \qquad k > 0 \tag{1.1}$$

1. $A > T => \frac{dT}{dt} > 0$: coffee is heating
2. $A < T => \frac{dT}{dt} < 0$: coffee is cooling
3. $A = T => \frac{dT}{dt} = 0$: equilibrium state

Solution of (1.1)*:*

$$\int \frac{dT}{A - T} = \int kdt$$
$$ln|T - A| = -kt + C_1$$
$$e^{ln|T-A|} = e^{-kt+C_1}$$

$$T - A > 0 => T(t) = A + C_2 e^(- kt)$$
$$T - A < 0 => T(t) = A - C_2 e^(- kt)$$
$$T - A = 0 => C_2 = 0$$

General solution of (1.1) is $T(t) = A + C_3 e^{-kt}$; $C_3 \in \mathbb{R}$
If $T_0 = T(0)$ is specified then $C_3 \equiv T_0 - A$, so that $T(t) = A + (T_0 - A)e^{-kt}$ solution of the initial value problem.

- k determines how fast $T(t)$ converges to A.

- Equation (1.1) is 1^{st} order, constant-coefficient, linear differential equation(DE). It is non-homogeneous or forced.
- t is independent, T is the dependent variable.
- Solution is reasonable as we would expect.

1.1.2 Application to "Time of Death": Coroner's Exam

- Death occurs at $t = 0$, $T(0) = 36°$
- Body temperature is measured at $t = t_1$ and $t = 1 + t_1 := t_2$, $T(t_1) = 25°$, $T(t_2) = 23°$, $A = 20°$
- Equation at $t_1 : 25 = 20 + (36 - 20)e^{-kt_1}$, hence $kt_1 = ln\frac{16}{5}$
- Equation at $t_2 : 23 = 20 + (36 - 20)e^{-kt_1}$, hence $kt_2 = ln\frac{16}{3}$

Using $t_2 = t_1 + 1$ and solving for k and t_1: $k = ln\frac{5}{3}$ $t_1 = \frac{ln(16/5)}{ln(5/3)}$, $t_1 = 2.28$ hours, death occured 2 hours 17 minutes before the first measurement. Two major assumptions made: A is constant and k is constant.

1.1.3 Newton's 2^{nd} Law of Motion

An object of mass m undergoes a net force $F(t)$. Let $x(t)$ be the displacement of the object from an equilibrium position $x(0) = x_0$.

$$m\frac{d^2x}{dt^2} = F(t) \tag{1.2}$$

Suppose $F(t) = F_0$, constant. Integrating once:

$$\begin{aligned} m\frac{dx}{dt} &= F_0 t + mv_0 \\ v_0 &= \left.\frac{dx}{dt}\right|_{t=0} \end{aligned}$$

and integrating once more:

$$\begin{aligned} mx(t) &= \frac{1}{2}F_0 t^2 + mv_0 t + mx_0 \\ x_0 &= x(0) \end{aligned}$$

Hence

$$\begin{aligned} x(t) = \frac{1}{2m}F_0 t^2 + v_0 t + x_0 \\ t \geq 0 \end{aligned}$$

with v_0:initial velocity and x_0: initial position

- (1.2) is a 2^{nd} order, linear, constant coefficient, non-homogeneous DE.
- It is not always possible to solve a differential equation by direct integration:

Example
$E(t) = L\frac{di}{dt} + v_c$

$$\frac{dE}{dt} = L\frac{d^2t}{dt^2} + \frac{1}{C}i \tag{1.3}$$

since $C\frac{dv_c}{dt} = i(t)$. If we try to solve (1.3) by integration, then

$$\int_0^t \frac{dE}{dt}dt = L\frac{di}{dt} + \frac{1}{2}\int_0^t i(t)dt + C_1$$

where both $\frac{di}{dt}$ and $\int i(t)dt$ are unknown and we can not proceed. We need another technique to solve (1.3), which is a $2^n d$ order, linear, constant-coefficient, non-homogeneous differential equation.

1.1.4 Population Dynamics

Let $N(t)$ be a population of something at time t. Then,

$$\begin{aligned} \frac{dN}{dt} &= (a - bN)N \\ N(0) &= N_0 \\ a, b &> 0 \end{aligned} \tag{1.4}$$

is called a "logistic population model". Its solution is,

$$N(t) = \frac{aN_0}{bN_0 + (a - bN_0)e^{-at}}$$

as can be checked by substituting into (1.4)

- Solution converges to $\frac{a}{b}$, starting at N_0 and converges monotonically increasing if $N_0 < \frac{a}{b}$
- (1.4) is nonlinear, 1^{st} order, homogeneous DE.
- if $N_0 > \frac{a}{b}$, then there is a vertical asymptote at $T = \frac{1}{a}ln(\frac{bN_0 - a}{bN_0})$, and population decreases to $\frac{a}{b}$

1.2 Ordinary Differential Equations

This Section can be found in the Section 2.2 of the Textbook

$$F(x, y(x), \dot{y}(x), \ldots, y^n(x)) = 0 \tag{1.5}$$

Is the most general ordinary differential equation since only the ordinary (not partial) derivatives of the dependent variable y appear in the argument of F. The function F has $n + 1$

arguments with x, the independent variable. This differential equation is linear of F is such that (1.5) can be rewritten as:

$$a_0(x)y^{(n)}(x) + \cdots + a_{n-1}(x)\dot{y}(x) + a_n y(x) = f(x) \tag{1.6}$$

for some functions $a_i(x)$ $(i = 0, \ldots, n)$ and $f(x)$. The linear differential equation (1.6) is homogeneous (unforced) if $f(x) \equiv 0$ otherwise, it is non-homogeneous (forced). The order of both (1.5) and (1.6) is n since this is the highest order derivative that appears (provided $a_0 \neq 0$).

Example

1. $\dot{y} + y^2 = 0$ is a 1^{st} order, nonlinear, ordinary differential equation.
2. $x^2\ddot{y} + y = 2sin(x)$ is a 2^{nd} order, linear, non-homogeneous differential equation.

A solution of (1.5) or (1.6) is a function of $y(x)$ that satisfies the equation for all x in interval $\mathbb{D} \subseteq \mathbb{R}$, which is also called a domain of validity.

Example

1. $y(x) = \frac{1}{x}$ is a solution in $\mathbb{D}_1 = (-\infty, 0)$ or $\mathbb{D}_2 = (0, \infty)$ of $\dot{y} + y^2 = 0$
2. $y(x) = Asin(x) + Bcos(x)$ is a solution to $\ddot{y} + y = 0$ for any $A, B \in \mathbb{R}$ in $\mathbb{D} = \mathbb{R} = (-\infty, \infty)$

A solution must be obviously be "differentiable" and hence "continuous" over the domain $\mathbb{D}$.

1.2.1 First Order Linear Differential Equations

$a_0(x)\dot{y} + a_1(x)y = f(x)$ can be put into the form:

$$\dot{y} + p(x)y = q(x) \tag{1.7}$$

$$p := \frac{a_1}{a_0}$$

$$q := \frac{f}{a_0}$$

Its homogeneous version is with $q(x) \equiv 0$:

$$\dot{y} + p(x)y = 0 \tag{1.8}$$

Which can be solved by direct integration:

$$
\begin{aligned}
\frac{dy}{dx} &= -p(x)y \\
\frac{dy}{y} &= -p(x)dx \\
\int \frac{dy}{y} &= -\int p(x)dx \\
ln|y| &= -\int p(x) + C \\
|y| &= e^{-\int p(x)dx} \cdot e^{C} \\
y(x) &= \pm Be^{-\int p(x)dx} \\
y(x) &= Ae^{-\int p(x)dx} \qquad (1.9)
\end{aligned}
$$

Where $A \in \mathbb{R}$, note that if $A \neq 0$, then $y(x) \neq 0$ so that our assumption above is satisfied. If $A = 0$, then (1.9) is still a solution to (1.8), i.e., the trivial solution $y(x) \equiv 0$. $\int p(x)dx$ in (1.9) is any arbitrary but fixed anti-derivative of $p(x)$ and exists whenever $p(x)$ is continuous, function of x.

Example

$\frac{dy}{dx} = -y$, then $p(x) = 1$, hence $\int p(x)dx = \int dx$ can be taken as x, $x+1$, $x-1$, etc. This gives alternative expressions for (1.9) as $y(x) = Ae^{-x}$ or Ae^{-x-1} or Ae^{-x+1}. But notice that A can take any value in $\mathbb{R}$ so that all these expressions are equivalent general solutions of (1.8).

1.2.2 Initial Value Problem

Suppose (1.8) is accompanied by the "initial condition" $y(a) = b$ for a value of a of x with $b \in \mathbb{R}$. Then,

$$y(x) = be^{\int_a^x p(\zeta)d\zeta} \qquad (1.10)$$

clearly satisfies $y(a) = b$, moreover b chain rule

$$\dot{y}(x) = b(-p(x))e^{\int_a^x p(\zeta)d\zeta} = -p(x)y(x)$$

so that $\dot{y} + py = 0$ is satisfied. Therefore, (1.10) is a solution to the initial value problem.

Example

$(x+2)\dot{y} - xy = 0$, $y(0) = 3$

$$
\begin{aligned}
\dot{y} - \frac{x}{x+2}y = 0 \\
p(x) &= -\frac{x}{x+2} \\
y(x) &= 3e^{\int_0^x(-\frac{\zeta}{\zeta+2})d\zeta} \\
&= 3e^x e^{-2ln|\zeta+2||_0^x} \\
&= 12e^x\frac{1}{(x+2)^2}
\end{aligned}
$$

Where $x \neq 2$, the solution is valid in $\mathbb{D}_1 = (-\infty, -2)$ or $\mathbb{D}_2 = (-2, \infty)$. Since the initial condition $a = 0$ is in $\mathbb{D}_2$, the domain of validity of the solution $y(x) = 12e^x(x+2)^{-2}$ is $\mathbb{D}_2$.

Let us now consider the non-homogeneous differential equation (1.7). Multiplying both sides by a yet unknown function $\sigma(x)$, called the integrating factor, we get:

$$\sigma\dot{y} + \sigma py = \sigma q \implies \frac{d}{dx}(\sigma y) - \dot{\sigma}y + \sigma py = \sigma q$$

The choice $\dot{\sigma} = \sigma p$ simplifies the equality to:

$$\frac{d}{dx}(\sigma y) = \sigma q \tag{1.11}$$

The equation $\dot{\sigma} - p\sigma = 0$ in unknown $\sigma(x)$ is a homogeneous 1^{st} order equation, one solution to which is:

$$\sigma(x) = e^{\int p(x)dx}$$

substituting in (1.11) and integrating, we have

$$
\begin{aligned}
\int \frac{d}{dx}(\sigma y)dx &= \int q(x)e^{\int p(x)dx}dx \\
\sigma(x)y(x) &= \int q(x)e^{\int p(x)dx}dx + C
\end{aligned}
$$

$$y(x) = e^{-\int p(x)dx}\left(\int q(x)e^{\int p(x)dx}dx + C\right) \tag{1.12}$$

This last equality gives the general solution y(x), in which the indefinite integral $\int p(x)dx$ appearing twice must be chosen consistently.

Example

$\dot{y} - 2xy = sinx$, $y(0) = 3 \implies p(x) = -2x$, $q(x) = sinx$

$$y(x) = e^{x^2}\left(\int_0^x sin\zeta e^{-\zeta^2}d\zeta + 3\right)$$

which satisfies $y(0) = 3$. As in this example, the solution of the initial value problem (1.7) together with $y(a) = b$ is

$$y(x) = e^{-\int_a^x p(\zeta)d\zeta}\left[\int_a^x q(\zeta)e^{\int_a^\zeta p(\mu)d\mu} + b\right] \tag{1.13}$$

- In (1.12), each value of the constant C corresponds to an initial condition y(a)=b, for some a and b.
- In order to be able to compute the integrals in (1.12) and (1.13), it is sufficient that $p(x)$ and $q(x)$ are continuous functions in an interval $\mathbb{I} \subseteq \mathbb{R}$. This is not necessary however.

Example

$x\dot{y} + 3y = 6x^3 \implies$ standard form $\dot{y} + \frac{3}{x}y = 6x^2$

$$\begin{aligned} y(x) &= e^{-\int \frac{3}{x}dx}\left[\int 6x^2 e^{\int \frac{3}{x}dx}dx + C\right] \\ &= e^{-3ln|x|}\left[\int 6x^2 e^{3ln|x|}dx + C\right] \\ &= \frac{1}{|x|^3}\left[\pm\int 6x^2|x|^3 dx + C\right] \\ &= \frac{1}{|x|^3}\left[\pm\int 6x^5 dx + C\right] \\ &= x^3 + \frac{B}{x^3} \end{aligned}$$

Where $B \in \mathbb{R}$, note that although $p(x) = \frac{3}{x}$ is not continuous at $x = 0$, there is still a solution satisfying $y(0) = 0$, which is $y(x) = x^3$. This is the limiting case for $B \to 0^+$ and $B \to 0^-$ in the figure below.

1.2.3 Variation of Parameter Method

The homogeneous equation (1.8) has the general solution

$$y(x) = Ae^{-\int p(x)dx}$$

For some $A \in \mathbb{R}$, method searches a solution for (1.7) by varying the parameter A with x. Consider

$$y(x) = Ae^{-\int p(x)dx}$$

as a possible solution for (1.7). Then,

$$\begin{aligned} \dot{y} &= \dot{A}e^{-\int p(x)dx} - Ape^{-p(x)dx} \\ &= \dot{A}e^{\int p(x)dx} - py \\ \dot{y} + py &= \dot{A}e^{-\int p(x)dx} = q \end{aligned}$$

Then,

$$\dot{A} = qe^{\int p(x)dx}$$

$$A = \int q(x)e^{p(x)dx}dx + C$$

which results in $y(x) = e^{-\int p(x)dx}\left(\int q(x)e^{\int p(x)dx}dx + C\right)$, this method also known as method of Lagrange.

1.3 First Order General Differential Equations

$$F(x, \dot{y}, y) = 0 \implies \dot{y} = f(x, y)$$

is possible of $F = 0$ can be solved for$\dot{y}$. Suppose f is a complicated function of x and y so that solution to $\dot{y} = f(x, y)$ is not obvious. We can try to get some idea about its solution by the concept of "direction field".
Direction field is the set of all "lineal elements", short straight lines of slopes $f(x_0, y_0)$ at the point (x_0, y_0), as (x_0, y_0) ranges through all possible values in xy-plane.

Example

$Ri(t) + L\frac{di}{dt} = E_0$; $i(0) = i_0$

$$\frac{di}{dt} + \frac{R}{L}i(t) = \frac{E_0}{L} \implies i(t) = e^{-\frac{R}{L}t}i_0 + \frac{E_0}{L}e^{-\frac{R}{L}t}\int_0^t e^{\frac{R}{L}\tau d\tau}$$

$$i(t) = \frac{E_0}{R} + (i_0 - \frac{E_0}{R})e^{-\frac{R}{L}t}$$

Suppose we could not obtain this solution but worked with

$$\frac{di}{dt} = f(t, i) = \frac{E_0}{L} - \frac{R}{L}i(t) \tag{1.14}$$

Direction field then would look like the dashed lines in(t, i) plane as shown. Along the line $i(t) = \frac{E_0}{R}$ all lineal elements would have slope zero as the right hand side of (1.14) is zero when $i = \frac{E_0}{R}$

1.3.1 More Applications

Radioactive Decay

Radioactive material decays proportional to its mass. If N is the number of nuclei, then $\frac{dN}{dt} = -kN$, $k > 0$ is a proportionality constant. Multiplying by "atomic mass", we obtain

$$\frac{dm}{dt} = -km(t)$$

$m(0) = m_0$ is the initial mass. Solution is $m(t) = m_0 e^{-kt}$ so that k is actually the "decay rate". In terms of $T =$ half life, which is defined by

$$\frac{1}{2}m_0 = m_0 e^{-kT} \implies kT = ln2$$

we can write

$$m(t) = m_0 e^{-\frac{ln2}{T}t} = m_0 2^{-\frac{t}{T}} \tag{1.15}$$

Carbon Dating Method

This method is used to determine the age of organic materials and it is based on (1.15). The element Carbon-14 is consumed by all animals and plants and has a half life if $T = 5,570$ years. A wood sample containing $0,2$ grams of C-14 today is t years old, where t satisfies

$$m(t) = 0.2 = m_0 e^{-t\frac{ln2}{5570}}$$

The amount m_0 of C-14 that wood sample contained when it died is found to be 2.6 grams. Hence, $0.2 = 2.6e^{-tln2/5570}$ or

$$t = \frac{ln13 \times 5570}{ln2} = 20,611\, years$$

Fluid Mixing Problems

A fluid containing a constant concentration c_1 of a solute comes in with a rate of $Q(t)\ cm^3/sec$. It goes out with the same rate $Q(t)$ and concentration $c(t)$, which is also the concentration in the tank. The volume v of the fluid in the tank remains, thus, constant. If $x(t)$ is the mass of the solute in the tank at time t, then

$$\frac{dx}{dt} = Q(t)c_1 - Q(t)c(t)$$

where $vc(t) = x(t)$, which results in

$$\frac{dx}{dt} + \frac{Q(t)}{v}x(t) = c_1 Q(t)$$

$c(0) = c_1$ results in $x(t) = c_1 v$, note that $x(t) = vc_1 = ct$ is a solution for any $Q(t)$.

1.3.2 First Order Separable Differential Equations

If $\dot{y} = f(x, y) = \mathfrak{X}(x)\mathfrak{Y}(y)$ for some $\mathfrak{X}$, $\mathfrak{Y}$ then it is separable. We can write:

$$\int \frac{dy}{\mathfrak{Y}(y)} = \int \mathfrak{X}(x)dx$$

Example

1. $x\dot{y} - y = (sin\dot{y})x \implies x = \frac{y}{\dot{y} - sin\dot{y}}$ Although x and y "separates", it is not separable.

2. $\dot{y} + y^2 = 0 \implies \int \frac{dy}{y^2} = -\int dx \implies -\frac{1}{y} + C = -x \implies y(x) = \frac{1}{1+C}$. If $y(0) = y_0$ is given, then $y(x) = \frac{y_0}{1+y_0 x}$ valid for $\mathbb{D}_1 = (-\frac{1}{y_0}, \infty)$ or for $\mathbb{D}_2 = (-\infty, -\frac{1}{y_0})$. If $y_0 > 0$, then the domain of validity is $\mathbb{D}_1$ and if $y_0 < 0$, then $\mathbb{D}_2$

3. $\dot{y} = \frac{4x}{1+2e^y}$, $y(0) = 1 \implies \int (1 + 2e^y)dy = \int 4xdx \implies y + 2e^y = 2x^2 + 1 + 2e$ is an implicit solution for y.

4.

$$
\begin{aligned}
doty = \frac{y(y-2)}{x(y-1)} &\implies \int \frac{y-1}{y(y-2)} dy = \int \frac{1}{x} dx \\
&\implies \int \left[\frac{1}{2y} + \frac{1}{2(y-2)}\right] dy = ln|x| + C \\
&\implies ln\frac{|y(y-2)|}{x^2} = 2C \\
&\implies y(y-2) = Ax^2 \\
&\implies y = 1 + \pm\sqrt{1 + Ax^2}\ ,\ A \in \mathbb{R}
\end{aligned}
$$

Let us consider some initial conditions:

- $y(0) = 0 \implies A$ is arbitrary and '-' sign holds $\implies y(x) = 1 - \sqrt{1 + Ax^2}$
- $y(0) = 2 \implies A$ is arbitrary and '+' sign holds $\implies y(x) = 1 + \sqrt{1 + Ax^2}$

Note that there are infinitely many functions satisfying ant of these two initial conditions. The functions are either half ellipses or half hyperbolas that change according to $A < 0$ or $A > 0$. $y(a) = 1 \implies \dot{y}$ is undefined for $a \in \mathbb{R}$. Note that any curve $(y-1)^2 + Ax^2 = 1$ would go through $y(a) = 1$ provided $A = \sqrt{\frac{1}{|a|}} > 0$. These are half ellipses with legs on the line $y = 1$. Since $\dot{y}$ is infinity at a, such functions do not really satisfy the differential equation. Hence, no solutions for initial values $y(a) = 1$. If $x \neq 0$ and $y \neq 1$, then there are unique solutions for such initial values. For instance,

- $y(1) = 4 \implies 3 = \pm\sqrt{1 + A} \implies A = 8,\ y = 1 + \sqrt{1 + 8x^2}$, upper part of the hyperbola $(y-1)^2 - \frac{x^2}{1/8} = 1$
- $y(1) = -3 \implies -4 = \pm\sqrt{1 + A} \implies A = 15,\ y = 1 - \sqrt{1 + 15x^2}$, lower part of the hyperbola $(y-1)^2 - \frac{x^2}{1/15} = 1$
- $y(1) = \frac{1}{2} \implies -\frac{1}{2} = \pm\sqrt{1 + A} \implies A = -3/4,\ y = 1 - \sqrt{1 - \frac{3}{4}x^2},\ -\frac{2}{\sqrt{3}} < x < \frac{2}{\sqrt{3}}$ lower part of the ellipse $(y-1)^2 - \frac{x^2}{16/9} = 1$

Example

$\frac{dN}{dt} = (a - bN)N$, $N(0) = N_0$, $b > 0$, $a \in \mathbb{R}$, (Logistic Equation)

$$\begin{aligned}
\int \frac{dN}{N(a-bN)} = \int dt \implies & \int \left[\frac{1}{aN} - \frac{1}{a}\frac{1}{N - a/b}\right] dN = t + C \\
\implies & \frac{1}{a}\left(ln|N| - ln\left|N - \frac{a}{b}\right|\right) = t + C \\
\implies & \left|\frac{N}{N - \frac{a}{b}}\right| = e^{at}e^{aC} \\
\implies & N = \frac{A\frac{a}{b}e^{at}}{Ae^{at} - 1} \\
\implies & N_0 = \frac{Aa}{b(A-1)} \implies A = \frac{bN_0}{bN_0 - a} \\
\implies & N = \frac{aN_0}{bN_0 + (a - bN_0)e^{-at}},\ t > 0
\end{aligned}$$

1.3.3 Exact Differential Equations

Recall partial derivation notations: $\dfrac{\partial f}{\partial x} = f_x$, $\dfrac{\partial^2 f}{\partial x \partial y} = f_{yx}$

Fact: If f_x, f_y, f_{xy}, f_{yx} are all continuous function in a region in xy-plane, then $f_{xy} = f_{yx}$, the order of differentiation does not matter. The differential df of a function $f(x, y)$ of two variables is defined by

$$df := f_x dx + f_y dy$$

which is a 1^{st} order approximation to $f(x + dx, y + dy) - f(x, y)$

Example

$$\frac{dy}{dx} = \frac{siny}{2y - xcosy} \implies siny\ dx + (x\ cosy - 2y)dy = 0$$

Notice that left hand side is dF for $F(x, y) = x\ siny - y^2$,

$$dF = siny\ dx + (x\ cosy - 2y)dy = \frac{\partial}{\partial x}(x\ siny - y^2)dx + \frac{\partial}{\partial y}(x\ siny - y^2)dy$$

The equation says $dF = 0$ and integration gives

$$F(x, y) = \int dF = C\ or\ x\ siny - y^2 = C;\ c \in \mathbb{R}$$

This constitutes an implicit solution to the differential equation.An initial condition $y(a) = b$ will fix $C = a\ sinb - b^2$. We generalize this procedure in the previous example. Consider a differential equation

$$\frac{dy}{dx} = -\frac{M(x, y)}{N(x, y)}$$

for some functions M, N that are continuous in a rectangle R, in xy-plane. Rewriting, we have

$$M(x, y)dx + N(x, y)dy = 0 \tag{1.16}$$

In order for (1.16) to be in the form $dF = 0$ for some function $F(x, y)$, we need

$$F_x = M,\ F_y = N \tag{1.17}$$

If (1.17) holds, then by continuity of M, N, therefore of F_x, F_y, we have,

$$F_{xy} = M_y = F_{yx} = N_x \implies M_y = N_x \tag{1.18}$$

Provided M_y, N_x are also continuous in R so that F_{xy}, F_{yx} are continuous. Conversely, if (1.18) holds, $M_y = N_x$ in R, then for F satisfying (1.17), we will have $dF = 0$.

Theorem: If M, N, M_y, N_x are continuous in a rectangle R in xy-plane, then $Mdx + Ndy$ is exact differential equation, there is $F(x, y)$ such that (1.17) holds in R, if and only if

$$M_y = N_x,\ \forall\,(x, y) \in R \tag{1.19}$$

Example

Previous example $siny\,dx + (x\,cosy - 2y)dy$ has $M = siny$ and $N = x\,cosy - 2y$ and (1.19) is satisfied since

$$M_y = N_x = cosy$$

To find a suitable $F(x, y)$, we can start by $F_x = M = siny$ to obtain $F(x, y) = x\,siny + A(y)$, by integrating keeping y a constant. The second condition in (1.17) is $F_y = N$, which is

$$N = x\,cosy - 2y = \frac{\partial}{\partial y}[x\,siny + A(y)] = x\,cosy + \dot{A}(y)$$

which gives $\dot{A}(y) = -2y$ or $A(y) = -y^2 + B$ for any $B \in \mathbb{R}$. Hence, $F(x, y) = x\,siny - y^2 + B$ for any B is a differential satisfying $dF = 0$. Therefore

$$F(x, y) = x\,siny - y^2 + B = C,\ C \in \mathbb{R}$$

or $x\,siny - y^2 = K$ for $K \in \mathbb{R}$, is an implicit solution of the original differential equation $\frac{dy}{dx} = \frac{siny}{2y - x\,cosy}$

Remark: Not all differential equations are exact since (1.19) can fail to hold: $dy/dx = sinx/[2y - x\,cosy] \implies M = sinx,\ N = y - cosx$ and $M_y = 0,\ N_x = sinx$. The condition (1.19) holds only along the line $x = 1$ in xy-plane.

Integrator Factor for Exactness

It can be shown that give $Mdx + Ndy = 0$ is not exact, one can always find $\sigma(x, y)$ such that

$$\sigma M dx + \sigma N dy = 0 \tag{1.20}$$

is exact. Of course, (1.20) would be exact if and only if

$$(\sigma M)_y = \sigma_y M + \sigma M_y = (\sigma N)_x = \sigma_x N + \sigma N_x$$

Suppose $\sigma = \sigma(x)$, (σ is independent of y), then $\sigma_y = 0$ and we have $\sigma M_y = \sigma_x N + \sigma N_x$ which gives

$$\frac{\sigma_x}{\sigma} = \frac{M_y - N_x}{N}$$

Thus, if $(M_y - N_x)/N$ is a function of x alone, then

$$ln|\sigma| = \int \frac{M_y - N_x}{N} dx \implies \sigma = e^{\frac{M_y - N_x}{N}} dx$$

is and integrating factor that makes (1.20) exact.
Suppose $\sigma = \sigma(y)$, (σ is independent of x), then $\sigma_x = 0$ and we have $\sigma_y M + \sigma M_y = \sigma N_x$ which gives

$$\frac{\sigma_y}{\sigma} = -\frac{M_y - N_x}{M}$$

Thus, if $(M_y - N_x)/M$ is a function of y alone, then

$$ln|\sigma| = -\int \frac{M_y - N_x}{M} dx \implies \sigma = e^{-\frac{M_y - N_x}{M}} dx$$

is and integrating factor that makes (1.20) exact.
Example

$$\frac{dy}{dx} = -\frac{1}{3x - e^{-2y}} \implies dx + (3x - e^{-2y})dy = 0$$

Since $M = 1$, $N = 3x - e^{-2y}$, $M_y = 0$, $N_x = 3 \neq M_y$, the equation is not exact. Note that $(M_y - N_x)/M = -3$ is independent of x. Hence,

$$\sigma(y) = e^{-\int \frac{M_y - N_x}{M} dy} = e^{-\int (-3) dy} = e^{3y}$$

is an integrating factor, i.e.,

$$e^{3y} dx + e^{3y}(3x - e^{-2y})dy = 0 \implies e^{3y} dx + (3xe^{3y} - e^{y})dy = 0$$

is exact. We have, in the searching for $F(x, y)$,

$$F_x = e^{3y}, \; F_y = 3xe^{3y} - e^{y}$$

so that $F(x, y) = xe^{3y} + A(y)$ for some $A(y)$ and

$$F_y = 3xe^{3y} + A'(y) = 3xe^{3y} - e^{y} \implies A'(y) = -e^{y} \implies A(y) = -e^{y} + B, \; B \in \mathbb{R}$$

This gives $F(x, y) = xe^{3y} - e^{y} + B$ so that, $dF = 0$ used, results in $F(x, y) = xe^{3y} - e^{y} + B = C$ or

$$xe^{3y} - e^{y} = K \text{ for } K \in \mathbb{R}$$

is general solution. This is again an "implicit solution".

Exercise: Obtain yet another method of solving 1^{st} order differential equation $\dot{y} + p(x) = q(x)$ by finding a suitable integrating factor for $(p(x)y - q(x))dx + dy = 0$

1.3.4 Linear Independence of Functions and Solutions of Differential Equations

A set of functions $\{u_1(x), \dots, u_k(x)\}$ is linearly dependent on an interval $\mathbb{I} \subseteq \mathbb{R}$ if at least one, say $u_1(x)$, can be written as

$$u_1(x) = \sum_{j=2}^{k} \alpha_j u_j(x) \text{ for some } \alpha_j \in \mathbb{R}$$

Otherwise, it is linearly independent and none can be expressed as a linear combination of others. Note that the zero-function can not be a member of any linearly independent set.

Example: $\{e^x, x^2, e^{-x}, \cosh(x)\}$ is linearly dependent on $\mathbb{R}$ since $\cosh(x) = \frac{e^x + e^{-x}}{2}$ $\forall x \in \mathbb{R}$. The sets $\{e^x, e^{-x}\}$, $\{e^x, \cosh(x)\}$, $\{e^{-x}, x^2, \cosh(x)\}$ are linearly independent.

Fact: $\{u_1, \dots, u_k\}$ is a linearly independent set on $\mathbb{I}$ if and only if

$$\sum_{j=1}^{k} \alpha_j u_j(x) = 0, \ \forall x \in \mathbb{I} \implies \alpha_j = 0 \text{ for } j = 1, \dots, k$$

Proof: If linearly independent but there are α_j's, not all zero such that $\sum_j \alpha_j u_j(x) = 0$, then u_k with $\alpha_k \neq 0$ can be written as a linear combination of others. Thus we have a contradiction. If linearly dependent, then there exist u_k such that $u_x - \sum_{j \neq k} \alpha_j u_j = 0$, which means there is a set $\{\alpha_1, \dots, \alpha_{j-1}, 1, \alpha_{j+1}, \dots, \alpha_k\}$, not all zero, with $\sum_j \alpha_j u_j = 0$

Example:$\{1, x, x^2, \dots, x^{n-1}\}$ is a linearly independent set on $\mathbb{R}$ for any $n \geq 1$. In fact, if for some α_j's, we have

$$\alpha_1 + \alpha_2 x + \cdots + \alpha_n x^{n-1} = 0$$

then by evaluating at $x = 0$, $x = 1$, ..., $x = n - 1$, we get.

$$\begin{bmatrix} 1 & 0 & \dots & 0 \\ 1 & 1 & \dots & 1 \\ 1 & 2 & \dots & 2^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & n-1 & \dots & (n-1)^{n-1} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \vdots \\ \alpha_n \end{bmatrix} = \underline{0}$$

The coefficient matrix is a Vandermonde matrix and it can be shown to be non-singular (invertible) so that $\alpha_j = 0$ for $j = 1, \dots, n$. Alternatively, we can take successive derivation of $\alpha_1 + \alpha_2 x + \cdots + \alpha_n^{n-1} = 0$ to obtain

$$\begin{bmatrix} 1 & x & x^2 & \dots & x^{n-1} \\ 0 & 1 & 2x & \dots & (n-1)x^{n-2} \\ 0 & 0 & 2 & \dots & (n-1)(n-2)x^{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & (n-1)! \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \vdots \\ \alpha_n \end{bmatrix} = \underline{0}$$

The coefficient matrix of which again non-singular. This gives the same result that α_j's are all zero. The fact on this page, then gives that $\{1, x, \ldots, x^{n-1}\}$ is linearly independent.

The second method above can be generalized: Give $\{u_1, \ldots, u_k\}$, suppose α_j exist such that $\sum_j \alpha_j u_j(x) = 0$ in an interval $\mathbb{I} \in \mathbb{R}$. Then, taking successive derivatives

$$\begin{bmatrix} u_1(x) & \ldots & u_k(x) \\ \dot{u}_1(x) & \ldots & \dot{u}_k(x) \\ \vdots & \ddots & \vdots \\ u_1^{(n-1)}(x) & \ldots & u_k^{(n-1)}(x) \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} = \underline{0}$$

If $\det \hat{W}[u_1, \ldots, u_k] =: W[u_1, \ldots, u_k]$, where $\hat{W}$ is the coefficient matrix, is non-zero at some $x \in \mathbb{I}$, then all α_j's should be zero, implying that $\{u_1, \ldots, u_k\}$ is linearly independent on $\mathbb{I}$. The converse is not true in general but it us true if $\{u_1, \ldots, u_k\}$ come from a solution to a differential equation (linear).

Theorem: If $u_1, \ldots, u_n$ are solutions to an n^{th} order homogeneous differential equation

$$\frac{d^n y}{dx^n} + p_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + p_{n-1}(x)\frac{dy}{dx} + p_n(x)y = 0 \tag{1.21}$$

where $p_j(x)$ are continuous on some $\mathbb{I} \in \mathbb{R}$, then $\{u_1, \ldots, u_n\}$ is linearly independent on $\mathbb{I}$ if and only if $W[u_1, \ldots, u_n](x) \neq 0$ for some $x \in \mathbb{I}$.

Example: $y''' - y' = 0$ admits solutions $u_1 = 1$, $u_2 = e^x$, $u_3 = e^{-x}$. We have

$$W[1, e^x, e^{-1}](x) = \det \begin{bmatrix} 1 & e^x & e^{-x} \\ 0 & e^x & -e^{-x} \\ 0 & e^x & e^{-x} \end{bmatrix} = 2$$

for any $x \in \mathbb{R}$. Hence, by the theorem above, $\{1, e^x, e^{-x}\}$ is linearly independent on $\mathbb{R}$

Example: Consider

$$u_1 := \begin{cases} x^2 & , x \leq 0 \\ 0 & , x \geq 0 \end{cases}, \ u_2 := \begin{cases} 0 & , x \leq 0 \\ x^2 & , x \geq 0 \end{cases}$$

so that

$$W[u_1, u_2](x) = \left\{ \begin{array}{cc} 0 & , x \leq 0 \\ 0 & , x \geq 0 \end{array} \right\} = 0, \ \forall x$$

However $\{u_1, u_2\}$ is obviously linearly independent on $\mathbb{R}$.

Remarks

(1) $W[u_1, \ldots, u_n](x)$ is called the Wronskian of $u_1, \ldots, u_n$. It is either zero on the whole of $\mathbb{I}$, provided $u_1, \ldots, u_n$are solutions of (1.21) on $\mathbb{I}$

(2) If $\{u_1, \ldots, u_k\}$ is a linearly independent set on $\mathbb{I}$, then

$$a_1 u_1(x) + \cdots + a_k u_k(x) = b_1 u_1(x) + \cdots + b_k u_k(x)$$

holds on $\mathbb{I}$ if and only if $a_j = b_j$ for $j = 1, \ldots, k$

Theorem (Existence and Uniqueness of Solution to (1.21)):If $p_1(x), \ldots, p_n(x)$ are continuous on a closed interval $\mathbb{I} \subseteq \mathbb{R}$, then the initial value problem, (1.21) together with

$$y(a) = b_1,\ \dot{y}(a) = b_2,\ \ldots,\ y^{(n-1)}(a) = b_n$$

for some $a \in \mathbb{R}$, $b_j \in \mathbb{R}$ has a unique solution $y(x)$ valid on $\mathbb{I}$

The differential operator

$$L := \frac{d^n}{dx^n} + p_1(x)\frac{d^{n-1}}{dx^{n-1}} + \cdots + p_{n-1}(x)\frac{d}{dx} + p_n(x)$$

allows us to shorten (1.21) to

$$L[y] = 0$$

This operator is linear since for any $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$,

$$L[\alpha_1 y_1 + \cdots + \alpha_n y_n] = \alpha_1 L[y_1] + \cdots + \alpha_n L[y_n]$$

for every $\{y_1, \ldots, y_n\}$

Corollary: If $y_1, \ldots, y_k$ are solution of (1.21), then so is $y = c_1 y_1 + \cdots + c_k y_k$ for any $c_1, \ldots, c_k \in \mathbb{R}$

Proof: $L[y] = L[c_1 y_1 + \cdots + c_k y_k] = c_1 L[y_1] + \cdots + c_n L[y_k] = 0$, since $L[y_j] = 0, \forall j = 1, \ldots, k$

Examples

(1) $y'' - 9y = 0$ has solutions $y_1 = e^{3x}$ and $y_2 = e^{-3x}$ so that $c_1 e^{3x} + c_2 e^{-3x}$ are solutions as well for any $c_1,\ c_2$.

(2) $x^3 y'' - yy' = 0$ has solutions $y_1 = 1$, $y_2 = x^2$. The linear combination $y = 4 + 3x^2$ is not a solution because differential equation is not linear.

(3) $y'' - 9y = 18$ has solutions $y_1 = 4e^{3x} - 2$, $y_2 = e^{3x} - 2$ but the linear combination $y = 5e^{3x} - 4$ is not. Superposition fails because differential equation is not homogeneous.

Theorem: Let $p_j(x)$ be continuous on an open interval $\mathbb{I} \subseteq \mathbb{R}$ for $j = 1, \ldots, n$. The homogeneous equation (1.21) has n linearly independent solutions $y_1(x), \ldots, y_n(x)$ on $\mathbb{I}$ and the general solution is

$$y(x) = c_1 y_1(x) + \cdots + c_n y_n(x) \tag{1.22}$$

for arbitrary constants $c_j \in \mathbb{R}$. If among $y_j(x)$, some are complex valued, then the corresponding c_j's are complex numbers.

Proof: By the existence and uniqueness of the solution to an initial value problem, there are n different solutions $y_1(x), \ldots, y_n x$ satisfying the following n sets of conditions

$$\begin{gathered} y_1(a) = 1,\ y_1'(a) = 0,\ \ldots,\ y_1^{(n-1)} = 0 \\ \vdots \\ y_n(a) = 1,\ y_n'(a) = 0,\ \ldots,\ y_n^{(n-1)} = 1 \end{gathered}$$

for some $a \in \mathbb{I}$ where in each row one value is 1 and all others are zero. The set $\{y_1(x), \ldots, y_n(x)\}$ is linearly independent since $W[y_1(x), \ldots, y_n(x)] = 1 \neq 0$. By superposition, any $y(x)$ as in (1.22), is a solution as well. We now show that there can be no other solution than (1.22). Suppose $y_{n+1}(x)$ is a solution that can not be written as in (1.22). Such a $y_{n+1}(x)$ must be linearly independent of $y_1(x), \ldots, y_n(x)$. However, let us consider for $\zeta \in \mathbb{I}$, the set of equations in $n+1$ unknowns $c_1, \ldots, c_n$:

$$n \text{ equations} \begin{cases} c_1 y_1(\zeta) + & \cdots + \quad c_{n+1} y_{n+1}(\zeta) = 0 \\ \quad\vdots & \qquad\qquad \vdots \\ c_1 y_1^{(n-1)}(\zeta) + & \cdots + \quad c_{n+1} y_{n+1}^{(n-1)} = 0, \quad \zeta \in \mathbb{I}(\zeta) = 0 \end{cases}$$

which has solution $c_1, \ldots, c_{n+1}$ not all zero, since there are only n equations for $n+1$ unknowns. Then,

$$v(x) = c_1 y_1(x) + \cdots + c_{n+1} y_{n+1}(x)$$

is a non-zero function satisfying $L[v] = 0$. Moreover,

$$v(\zeta) = 0,\ v'(\zeta) = 0,\ \ldots\ , v^{(n-1)}(\zeta) = 0$$

so that $v(x)$ must be the zero function on $\mathbb{I}$, by uniqueness of the solution to an initial value problem. This shows that $\{y_1, \ldots, y_{n+1}\}$ is a linearly independent set, which is a contradiction.

- Any set of n linearly independent solutions of (1.21) is called a basis of solutions of (1.21)

  Example: $\{e^{3x}, \sinh(3x)\}$, $\{e^{3x}, e^{-3x}\}$, $\{e^{3x}, \cosh(3x)\}$ are all bases for the solutions of $y'' - 9y = 0$ on $\mathbb{R}$ since each is linearly independent and each contains $n = 2$ functions that satisfy $y'' - 9y = 0$

- It is clear that initial value problem can be solved using the general solution. Consider $y''' + y' = 0$ subject to $y(0) = 3$, $y'(0) = 5$, $y''(0) = -4$. The general solution is

  $$y(x) = c_1 \cos(x) + c_2 \sin(x) + c_3;\ c_1, c_2, c_3 \in \mathbb{R}$$

  since $\{1, \cos(x), \sin(x)\}$ is a basis. Using the initial conditions

  $$\begin{aligned} y(0) &= c_1 + c_3 = 3 \\ y'(0) &= -c_1 \sin(0) + c_2 \cos(0) = c_2 = 5 \\ y''(0) &= -c_1 \cos(0) - c_2 \sin(0) = -c_1 = -4 \end{aligned}$$

  so that $y(x) = 4\cos(x) + 5\sin(x) - 1$ solves the initial value problem.

- **Boundary Value Problems:** If conditions are specified for mixed points in $\mathbb{I} \in \mathbb{R}$, then we have a "boundary value problem". Such problems can have no solution, a unique solutions, or many solutions.

Example: $y'' + y = 0$ has the general solution on $\mathbb{I} \in \mathbb{R}$

$$y(x) = c_1 \cos(x) + c_2 \sin(x);\ c_1, c_2 \in \mathbb{R}$$

(1) $y(0) = 2,\ y(\pi) = 1 \implies c_1 = 2,\ c_1 = -1 \implies$ No solution.

(2) $y(0) = 2,\ y(\pi/2) = 3 \implies c_1 = 2,\ c_2 = 3 \implies$ Unique Solution

(3) $y(0) = 2,\ y(\pi) = -1 \implies c_1 = 2,\ c_2$ is free $\implies$ Infinitely many solutions.

1.3.5 A Short Review of Complex Calculus

Fact 1: (Fundamental Theorem of Algebra) Any polynomial equation $\lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n = 0$ for $a_j \in \mathbb{R}$, has exactly n solutions $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ (some of which may be repeated)

Example: $(\lambda_2 + 1)^2 = \lambda^4 + 2\lambda^2 + 1 = 0$ has 4 solutions

$$\lambda_1 = i,\ \lambda_2 = i,\ \lambda_3 = -i,\ \lambda_4 = -i$$

Definition: Euler's Formula: $\forall x, y \in \mathbb{R}$

$$e^{x+iy} = e^x(\cos(y) + i\sin(y))$$

This definition make sense because

$$\begin{aligned} e^{x+iy} &= e^x e^{iy} \\ &= e^x \sum_{k=0}^{\infty} \frac{(iy)^k}{k!} \\ &= e^x \left(\sum_{k=0,\text{ k odd}}^{\infty} \frac{(-1)^{k/2} y^k}{k!} + i \sum_{k=1,\text{ k even}}^{\infty} \frac{(-1)^{(k-1)/2} y^k}{k!} \right) \\ &= e^x(\cos(y) + i\sin(y)) \end{aligned}$$

Fact 2: $e^{-iy} = \cos(y) - i\sin(y),\ \cos(y) = \dfrac{e^{iy} + e^{-iy}}{2},\ \sin(y) = \dfrac{e^{iy} - e^{-iy}}{2i}$ *Fact 3:* If $\lambda = a + ib;\ a, b \in \mathbb{R}$, then $\forall x \in \mathbb{R}$

$$\frac{d}{dx} e^{\lambda x} = \lambda e^{\lambda x}$$

Definition: $\cos(\lambda x) = \dfrac{e^{i\lambda x} + e^{-i\lambda x}}{2},\ \sin(\lambda x) = \dfrac{e^{i\lambda x} - e^{-i\lambda x}}{2i}$ for $\lambda \in \mathbb{C}$ and $x \in \mathbb{R}$

Fact 4:

$$\frac{d}{dx}(\cos(\lambda x)) = -\lambda \sin(\lambda x),\ \frac{d}{dx}(\sin(\lambda x)) = \lambda \cos(\lambda x)$$

1.4 Second Order Linear Homogeneous Differential Equations

$$y'' + a_1 y' + a_2 y = 0\ ;\ a_1, a_2 \in \mathbb{R} \tag{1.23}$$

Let us try (motivated by the solution e^{ax} of $y' - ay = 0$) the function $y^{\lambda x}$ as the candidate solution for $\lambda \in \mathbb{C}$. Substituting in (1.23), we have

$$\lambda^2 e^{\lambda x} + a_2 \lambda e^{\lambda x} + a_2 e^{\lambda x} = 0 \implies \lambda^2 + a_1 \lambda + a_2 = 0$$

the solution of which are

$$\lambda_{1,2} = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2}$$

If $a_1^2 - 4a_2 \neq 0$, then $\lambda_1 \neq \lambda_2$ and (possibly complex-valued functions) $e^{\lambda_1 x}$, $e^{\lambda_2 x}$ are solutions which are linearly independent as $W[e^{\lambda_1 x}, e^{\lambda_2 x}] \neq 0$. Hence, if $\lambda_1 \neq \lambda_2$, $\{e^{\lambda_1 x}, e^{\lambda_2 x}\}$ is a basis of the solution of (1.23). Note that if λ_1, λ_2 are complex (non-real), then $\lambda_1 = a + ib$, $\lambda_2 = a - ib$ so that $\{e^{ax}\cos(bx), e^{ax}\sin(bx)\}$ is another basis in the terms of real-valued functions.

Examples

(1) (Real $\lambda_1 \neq \lambda_2$) $y'' - 9y = 0 implies \lambda^2 - 9 = 0 \implies \lambda_{1,2} = \pm 3$

$$y(x) = c_1 e^{3x} + c_2 e^{-3x};\ c_1, c_2 \in \mathbb{R}$$

(2) (Imaginary $\lambda_1 \neq \lambda_2$) $y'' + 9y = 0 \implies \lambda^2 + 9 = 0 \implies \lambda = \pm 3i$. Hence $a = 0$, $b = 3$ and

$$y(x) = c_1 \cos(3x) + c_2 \sin(3x);\ c_1, c_2 \in \mathbb{R}$$

Note that if we used $\{e^{i3x}, e^{-i3x}\}$ as a basis, then the general solution is
$y(x) = c_1 e^{i3x} + c_2 e^{-i3x};\ c_1, c_2 \in \mathbb{R}!$

(3) (Non-real roots $\lambda_1 \neq \lambda_2$) $y'' + 4y' + 9y = 0 \implies \lambda^2 + 4\lambda + 9 = 0 \implies \lambda_{1,2} = -2 \pm i\sqrt{5}$, so that $a = -2$, $b = \sqrt{5}$ and

$$y(x) = e^{-2x}(c_1 \cos(\sqrt{5}x)) + c_2 \sin(\sqrt{5}x)$$

for $c_1, c_2 \in \mathbb{R}$ is the general solution.

(4) (Repeated Roots $\lambda_1 = \lambda_2$) Characteristic equation produces only one solution $y_1 = e^{\lambda_1 x}$. To find the second solution, we may try $y_2(x) = A(x)e^{\lambda_1 x}$ for some $A(x)$. Substituting into (1.23) gives

$$y'' + a_1 y' a_2 y = A'' e^{\lambda_1 x} + 2\lambda_1 A' e^{\lambda_1 x} + a_1(A' e^{\lambda_1 x} + \lambda_1 A e^{\lambda_1 x}) + A a_2 e^{\lambda_1 x} = 0$$

$$\implies \cancelto{0}{(\lambda_1^2 + a_1\lambda_1 + a_2)} A e^{\lambda_1 x} + \cancelto{0}{(2\lambda + a_1)} A' e^{\lambda_1 x} + A'' e^{\lambda_1 x}$$
$$\implies A'' = 0 \implies A(x) = Bx + C,\ B, C \in \mathbb{R}$$

Thus, $y_2(x) = x e^{\lambda_1 x}$ is a solution and is linearly independent of $e^{\lambda_1 x}$, i.e., $\{e^{\lambda_1 x}, x e^{\lambda_1 x}\}$ is a basis of solution.

Example: $y'' + 4y' + 4y = 0 \implies \lambda^2 + 4\lambda + 4 = 0 \implies \lambda_{1,2} = -2$

$$y(x) = c_1 e^{-2x} + c_2 x e^{-2x};\ c_1, c_2 \in \mathbb{R}$$

1.5 n-th Order Linear Constant Coefficient Differential Equations

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = 0; \; a_j \in \mathbb{R} \tag{1.24}$$

Substituting $e^{\lambda x}$ as a candidate solution, we find that

$$lambda^n + a_1 \lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n = 0$$

which has n solutions $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ some of which may be repeated and some non-real.

1.5.1 Case 1: n Distinct Roots

In this case there are n different solutions $e^{\lambda_j x}, \; j = 1, \ldots, n$

Fact: $\{e^{\lambda_1 x}, \ldots, e^{\lambda_n x}\}$ is linearly independent provided $\lambda_j \neq \lambda_k$ for every $j \neq k$

Proof: $W[e^{\lambda_1 x}, \ldots, e^{\lambda_n x}]$ is

$$\det \begin{bmatrix} e^{\lambda_1 x} & \ldots & e^{\lambda_n x} \\ \lambda_1 e^{\lambda_1 x} & \ldots & \lambda_n e^{\lambda_n x} \\ \vdots & \ddots & \vdots \\ \lambda_n^{n-1} e^{\lambda_1 x} & \ldots & \lambda_n^{n-1} e^{\lambda_n x} \end{bmatrix} = (-1)^{\dfrac{n(n-1)}{2}} e^{-a_1 x} \prod_{j<k} (\lambda_j - \lambda_k)$$

When one uses $-a_1 = \lambda_1 + \cdots + \lambda_n$ and the Vandermonde determinant formula.

$$\det \begin{bmatrix} 1 & \ldots & 1 \\ \lambda_1 & \ldots & \lambda_n \\ \vdots & \ddots & \vdots \\ \lambda_1^{n-1} & \ldots & \lambda_n^{n-1} \end{bmatrix} = (-1)^{\dfrac{n(n-1)}{2}} \prod_{j<k} (\lambda_j - \lambda_k)$$

Examples

(1) $y^{(5)} - 7y^{(3)} + 12y' = 0 \implies \lambda(\lambda^4 - 7\lambda^2 + 12) = 0 \implies \lambda_1 = 0, \; \lambda_2 = \sqrt{3}, \; \lambda_3 = -\sqrt{3}, \; \lambda_4 = 2, \; \lambda_5 = -2$

$$y(x) = c_1 + c_2 e^{\sqrt{3}x} + c_3 e^{-\sqrt{3}x} + c_4 e^{2x} + c_4 e^{-2x}$$

for $c_1, c_2, c_3, c_4, c_5 \in \mathbb{R}$

(2) $y''' - y = 0 \implies \lambda_1 = 1, \; \lambda_{2,3} = \dfrac{-1 \pm i\sqrt{3}}{2}$. A basis of the solutions is $\{e^x, e^{-x/2}\cos(\sqrt{3}x/2), e^{-x/2}\sin(\sqrt{3}x/2)\}$ in terms of real valued functions, so that general solution is

$$y(x) = c_1 e^x + e^{\frac{-1}{2}x}(c_2 \cos(\frac{\sqrt{3}}{2}x) + c_3 \sin(\frac{\sqrt{3}}{2}x))$$

for $c_1, c_2, c_3 \in \mathbb{R}$

1.5.2 Case 2: k Distinct Roots

General solution for k distinct roots $\lambda_1, \dots, \lambda_k$ with multiplicities $m_1, \dots, m_k$ is

$$y(x) = \sum_{j=1}^{k} (c_{1j} e^{\lambda_j x} + \cdots + c_{m_j j} x^{m_j - 1} e^{\lambda_j x})$$

for c_{lj}'s among which those multiplying complex valued functions are non real, complex. Thus, for each λ_j with multiplicity m_j the corresponding basis functions are

$$\{e^{\lambda_j x}, x e^{\lambda_j x}, \dots, x^{m_j - 1} e^{\lambda_j x}\}$$

If λ_j is non real then its conjugate is also among $\lambda_1, \dots, \lambda_k$ so that two bases functions can be combined to take, with $\lambda_j = a_j + i b_j$

$$\{e^{a_j x} \cos(b_j x), e^{a_j x} \sin(b_j x), \dots, x^{m_j - 1 e^{a_j x} \cos(b_j x), x^{m_j - 1} e^{a_j x} \sin(b_j x)}\}$$

<u>**Examples**</u>

(1) $y^{(4)} - 8y''' + 26y'' - 40y' + 25y = 0 \implies (\lambda^2 - 4\lambda + 5)^2 \implies \lambda_1 = 2 + i,\ \lambda_2 = 2 - i,\ m_1 = m_2 = 2$

$$y(x) = e^{2x}[(A + Bx)\cos(x) + (C + Dx)\sin(x)];\ A, B, C, D \in \mathbb{R}$$

(2) $y^{(5)} - y^{(3)} = 0 \implies \lambda_1 = 0,\ m_1 = 3,\ \lambda_2 = 1, \lambda_3 = -1$ so that

$$y(x) = \underbrace{(c_1 + c_2 x + c_3 x^2)}_{\lambda_1 = 0} + c_4 e^x + c_5 e^{-x}$$

1.5.3 Homogeneous Cauchy-Euler Equation

Let $c_j \in \mathbb{R}$ for $j = 1, \dots, n$ and consider

$$x^n y^{(n)} + c_1 x^{n-1} y^{(n-1)} + \cdots + c_{n-1} x y' + c_n y = 0$$

In order to determine the general solution, we use a direct approach and consider the special case $n = 1$, $n = 2$ first.

n=1:

$$\begin{aligned} xy' + c_1 y = 0 &\implies \frac{dy}{dx} = -c_1 \frac{dx}{x} \\ &\implies \ln|y| = -c_1 \ln|x| + B \\ &\implies \ln|y| = \ln|x|^{-c_1} + \underbrace{\ln A}_{B} \\ &\implies |y| = A|x|^{-c_1},\ A \in \mathbb{R} \\ &\implies y = \begin{cases} Ax^{-c_1} & , x \geq 0 \\ -A(-x)^{-c_1} & , x \leq 0 \end{cases} \\ &\implies \boxed{y(x) = A|x|^{-c_1}},\ A \in \mathbb{R} \end{aligned} \tag{1.25}$$

n=2:

$x^2y'' + c_1xy' + c_2y = 0$. Suppose $x > 0$ for simplicity. Let us postulate that there is a solution like $y = x^\lambda$ for some $\lambda \in \mathbb{C}$. Then

$$\begin{aligned} & x^2(\lambda-1)\lambda x^{\lambda-2} + c_1x\lambda x^{\lambda-1} + c_2\lambda = 0 \\ \implies & (\lambda(\lambda-1) + c_1\lambda + c_2)x^\lambda = 0 \\ \implies & \lambda^2 + (c_1-1)\lambda + c_2 = 0 \text{ with the roots } \lambda_1, \lambda_2 \in \mathbb{C} \end{aligned}$$

Suppose first that $\lambda_1 \neq \lambda_2$: Then $y_1 = x^{\lambda_1}$, $y_2 = x^{\lambda_2}$ are two distinct solutions, which are linearly independent as

$$W[x^{\lambda_1}, x^{\lambda_2}](x) = det \begin{bmatrix} x^{\lambda_1} & x^{\lambda_2} \\ \lambda_1 x^{\lambda_1-1} & \lambda_2 x^{\lambda_2-1} \end{bmatrix} = (\lambda_2-\lambda_1)x^{\lambda_1+\lambda_2-1} \neq 0$$

The general solutions is thus

$$y(x) = Ax^{\lambda_1} + Bx^{\lambda_2};\ A, B \in \mathbb{C};\ x > 0$$

If $\lambda_1 = \lambda_2$, then any $y(x) = Ax^{\lambda_1}$ for $A \in \mathbb{R}$ is a solution. Let us again use the variation of parameter method and consider $y(x) = A(x)x^{\lambda_1}$ as a possible solution. Then

$$y' = A'x^{\lambda_1} + A\lambda_1 x^{\lambda_1-1},\ y'' = A''x^{\lambda_1} + 2A'\lambda_1 e^{\lambda_1-1} + A\lambda(\lambda-1)e^{\lambda-2}$$

so that

$$\begin{aligned} & x^2(A''x^{\lambda_1} + 2A'\lambda_1 x^{\lambda_1-1} + A\lambda_1(\lambda_1-1)e^{\lambda_1-2}) + c_1x(A'x^{\lambda_1} + A\lambda_1 e^{\lambda_1-1}) + c_2Ax^{\lambda_1} \\ & = A''x^{\lambda_1+2} + A'\cancelto{0}{(2\lambda_1+c_1)}x^{\lambda_1+1} + A\cancelto{0}{(\lambda_1(\lambda_1-1)+c_1\lambda_1+c_2)}x^{\lambda_1} = 0 \\ & \implies (xA'' + A')x^{\lambda_1+1} = 0 \\ & \implies xA'' + A' = 0 \\ & \implies xp' + p = 0;\ p = A', \\ & \implies p(x) = Ax^{-1} \text{ (from (1.25))} \\ & \implies A(x) = B\ln x + C \end{aligned}$$

for $B, C \in \mathbb{R}$. Therefore, for any B, C; a solution is $\boxed{y(x) = (B\ln x + C)x^{\lambda_1}}$

Remark: If $\lambda_1 = \bar{\lambda_2} = \alpha + i\beta \in \mathbb{C}$ with $\alpha, \beta \in \mathbb{R}$, then

$$x^{\lambda_1} = e^{\lambda_1 \ln x} = e^{\alpha \ln x}e^{i\beta \ln x} = x^\alpha(\cos(\beta \ln x))$$

so that the complex valued basis $\{x^{\lambda_1}, x^{\lambda_2}\}$ can be transformed to a real valued basis

$$\{x^\alpha \cos(\beta \ln x), x^\alpha \sin(\beta \ln x)\}$$

It follows that when roots of $\lambda^2 + (c_1-1)\lambda + c_2 = 0$ are complex conjugate, then the general solution for $x > 0$ is

$$y(x) = x^\alpha(A\cos(\beta \ln x) + B\sin(\beta \ln x));\ A, B \in \mathbb{R}$$

Example: $x^2y'' - xy' + 2y = 0 \implies \lambda(\lambda-1) - \lambda + 2 = 0 \implies \lambda_{1,2} = 1 \pm i$ so that $\alpha = 1$, $\beta = 1$. Hence,

$$y(x) = x[A\cos(\ln x) + B\sin(\ln x)]\ ;\ x > 0,\ A, B \in \mathbb{R}$$

Solution for $x < 0$

Substitute $t = -x$ and let

$$\begin{aligned} y(x) = y(-t) &= Y(t) \\ \frac{dy}{dx} = \frac{dY}{dt}\frac{dt}{dx} &= -\frac{dY}{dt} \\ \frac{d^2y}{dx^2} = \frac{d}{dt}\left(-\frac{dY}{dt}\right)\frac{dt}{dx} &= \frac{d^2Y}{dt^2} \end{aligned}$$

so that $x^2y'' + c_1xy' + c_2y = 0$ transforms to

$$t^2Y'' + c_1tY' + c_2Y$$

which has the basis $\{t^{\lambda_1}, t^{\lambda_2}\} = \{(-x)^{\lambda_1}, (-x)^{\lambda_2}\};\ \lambda_1 \neq \lambda_2$ and $\{t^{\lambda_1}, (\ln(t))t^{\lambda_1}\}$
$= \{(-x)^{\lambda_1}, (\ln(-x))^{\lambda_1}\};\ \lambda_1 = \lambda_2$

It follows that for $x \in \mathbb{R}$, we can write the solution as

$$y(x) = \begin{cases} A|x|^{\lambda_1} + B|x|^{\lambda_2} & , \lambda_1 \neq \lambda_2 \qquad\qquad , \lambda_1, \lambda_2 \text{ both real} \\ (A + B\ln|x|)|x|^{\lambda_1} & , \lambda_1 = \lambda_2 \\ |x|^\alpha[A\cos(\beta\ln|x|) + B\sin(\beta\ln|x|)] & , \lambda_1 = \bar{\lambda_1} = \alpha + i\beta \end{cases}$$

1.5.4 n^{th} Order Cauchy-Euler Equation

$$x^ny^{(n)} + c_1x^{n-1}y^{(n-1)} + \cdots + c_{n-1}xy' + c_ny = 0$$

form the characteristic equation in λ as

$$\lambda(\lambda-1)\ldots(\lambda-n+1) + c_1\lambda(\lambda-1)\ldots(\lambda-n+2) + \cdots + c_{n-1}\lambda + c_n = 0$$

If $\lambda_1, \ldots, \lambda_k$ are its distinct roots with multiplicities $m_1, \ldots, m_k$ respectively, then the general solution is

$$y(x) = \sum_{j=1}^{k}[A_{1j} + A_{2j}\ln|x| + \cdots + A_{m_jj}(\ln|x|)^{m_j-1}]|x|^{\lambda j}$$

for A_{kj}, which is non-real if and only if λ_j is non-real. Note that a non-real λ_j will have its conjugate $\bar{\lambda_j}$among the distinct roots and the two terms in the summation can be combined to obtain real-valued functions of x.

Example: $x^4y^{(4)} + 10x^3y^{(3)} + 33x^2y^{(2)} + 39xy' + 25y = 0$, characteristic equations is

$$\begin{aligned} \lambda(\lambda-1)(\lambda-2)(\lambda-3) + 10\lambda(\lambda-1)(\lambda-2) + 33\lambda(\lambda-1) + 39\lambda + 25 &= \lambda^4 + 4\lambda^3 + 14\lambda^2 + 20\lambda + 25 \\ &= (\lambda^2 + 2\lambda + 5)^2 \\ &= [(\lambda+1)^2 + 4]^2 \\ &= 0 \implies \lambda_1 = -1 + 2j = \bar{\lambda_2}, \\ m_1 = 2,\ m_2 = 2 & \end{aligned}$$

General solution:

$$y(x) = |x|^{-1}[(A + B\ln|x|)\cos(2\ln|x|) + (C + D\ln|x|)\sin(2\ln|x|)] \text{ for } A, B, C, D \in \mathbb{R}$$

Remark: Suppose $x = e^{\zeta}$ or $\zeta = \ln|x|$. Then $y(x) = y(e^{\zeta}) = Y(\zeta)$ so that

$$\frac{dY}{dx} = \frac{dY}{d\zeta}\frac{d\zeta}{dx} = Y'\frac{1}{x}, \; \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(Y'\frac{1}{x}\right)$$

$$\implies \frac{d^2y}{dx^2} = -\frac{1}{x^2}Y' + \frac{1}{x}Y''\frac{d\zeta}{dx} = \frac{1}{x^2}(Y'' - Y')$$

Hence, $x^2y'' + c_1xy' + c_2y = Y'' - Y' + c_1Y' + c_2Y = Y'' + (c_1 - 1)Y' + Y$ which has $\{e^{\lambda_1\zeta}, e^{\lambda_2\zeta}\} = \{x^{\lambda_1}, x^{\lambda_2}\}$ as its basis.

1.6 Non-Homogeneous Differential Equation

Let $L[\cdot]$ be the operator defined on 18 and consider

$$L[y]f(x); \; f(x) \text{ is a given funciton} \tag{1.26}$$

If $y_n(x)$ denotes the general solution of the homogeneous $L[y] = 0$ and if $y_p(x)$ is particular solution of (1.26), then $L[y_p] = f$ and

$$L[y_p + y_h] = L[y_p] + L[y_h] = L[y_p] = f$$

so that $y_p + y_h$ is also a solution of (1.26). Conversely, given any $y(x)$ satisfying (1.26), we have $L[y - y_p] = L[y] - L[y_p] = f - f = 0$ so that $y - y_p$ in a homogeneous solution, i.e., $y - y_p = y_h$ for some specialization of the general homogeneous solution. Therefore, $y_p + y_h$ is the general solution of (1.26). We can "split up" our search for a particular solution $y_p(x)$.

Fact: If $f = f_1 + \cdots + f_k$ for some functions $f_j(x)$, then the general solution of (1.26) can be expressed as

$$y = y_{p_1} + \cdots + y_{p_k} + y_h$$

where $y_{p_j}(x)$ is a particular solution to $L[y_{p_j}] = f_j(x)$, for $j = 1, \ldots, k$

Proof: $L[y_{p_1} + \cdots + y_{p_k}] = L[y_{p_1}] + \cdots + Ł[y_{p_k}] = f_1 + \cdots + f_k = f$

We will consider two alternative methods of finding y_p:

(i) The Method of Undetermined Coefficient

(ii) The Method of The Variation of Parameter

The first method applies to constant-coefficient linear differential equations, the second is general.

1.6.1 Undetermined Coefficients Method

This method can be used if

(a) L is a constant coefficient, and

(b) $f = f_1 + \cdots + f_k$ and the derivative of any order of each f_j is a linear combination of a finite number of linearly independent functions.

Example:

1) $f_j = 2xe^{-x} \implies \{f_j, f_j', f_j'', \dots\} = \{2xe^{-x}, \dots\}$ so that $f_j^{(l)}$ for any $l \geq 0$ is a linear combination of $\{e^{-x}, xe^{-x}\}$; two linearly independent functions.

2) $f_j = x^n + 1 \implies \{f_j, f_j', \dots, f_j^{(n)}\} = \{x^n + 1, nx^{n-1}, n(n-1)x^{n-2}, \dots, n!\}$ so that $\{1, x, \dots, x^{n+1}\}$ is a basis for the space of all derivatives.

3) $f_j = \frac{1}{x} \implies \{f_j, f_j', \dots\} = \{\frac{1}{x}, \frac{-1}{x^2}, \frac{2}{x^3}, \dots\}$ so that f_j does not satisfy the condition b

Claim: A function $f_j(x)$ satisfies condition b if and only if it is itself a solution of a linear, constant coefficient, homogeneous differential equation.

Proof: Condition b is equivalent to: There is $N \geq 0$ and constants $a_1, \dots, a_N$ such that f_j satisfies

$$a_0 f_j^{(N)} + a_1 f_j^{(N-1)} + \cdots + a_{N-1} f_j' + a_N f_j = 0,\ a_0 \neq 0 \tag{1.27}$$

because if $f_j^{(n)} = b_{1N} g_1(x) + \cdots + b_{Nn} g_N(x)$, for functions $g_j(x)$ that are linearly independent for any n, then for $n = 1, \dots, N$ we have

$$\begin{bmatrix} f_j \\ f_j' \\ \vdots \\ f_j^{(N)} \end{bmatrix} = \begin{bmatrix} b_{10} & \dots & b_{N0} \\ b_{11} & \dots & b_{N1} \\ \vdots & \ddots & \vdots \\ b_{1N} & \dots & b_{NN} \end{bmatrix} \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_N \end{bmatrix}$$

where the $(N+1) \times N$ matrix has linearly dependent rows. Hence, there are constants $a_0, a_1, \dots, a_N$ not all zero such that (1.27) holds. Conversely, if (1.27) holds, then with $g_1 = f_j,\ g_2 = f_j', \dots, g_N = f_j^{(N-1)}$, we have that any $f_j^{(N)}$ with $n \geq N$ can be expressed as a linear combination of g_j's. We show that how to apply the method of undetermined coefficients by an example

Example: Find the general solution of

$$y^{(4)} - y^{(2)} = -\sin(2x) + 3x^2$$

Let $f_1 = -\sin(2x),\ f_2 = 3x^2$, both of which satisfy b with $f_1 = -\sin(2x) \implies \{\sin(2x), \cos(2x)\}$, $\{f_2 = 3x^2 \implies \{1, x, x^2\}$.

$L[y] = f_1(x) = -\sin(2x)$: candidate solution is formed as

$$y_{p_1}(x) = A\sin(2x) + B\cos(2x);\ B, A \text{ to be found}$$

Taking derivatives and plugging the results then equating the coefficients will give $B = 0,\ A = -\frac{1}{20}$ so that $y_{p_1}(x) = -\dfrac{1}{20}\sin(2x)$.

$L[y] = f_2(x) = 3x^2$: Candidate solution is

$$y_{p_2}(x) = Ax^2 + Bx + C;\ A, B, C \text{ to be found.}$$

Taking derivatives and plugging the results would give us no solution for A, the reason why this happens is that some of the linearly independent functions $\{1, x, x^2\}$ also occurs in $y_h(x)$

$L[y_h] = 0$ (Homogeneous Solution): Characteristic equation $\lambda^3 - \lambda^2 = 0$ give $\lambda_1 = 0,\ \lambda_2 = 1, \lambda_3 = -1, m_1 = 2$ so that

$$y_h(x) = c_{11} + c_{21}x + c_{12}e^x + c_{13}e^{-x}$$

Note that $1, x \in \{1, x, x^2\}$. We now modify the candidate solution to $L[y] = f_2$ as

$$y_{p_2}(x) = Dx^4 + Ex^3 + Fx^2$$

which is obtained by multiplying the previous candidate by the smallest power of x (x^2 in this case) such that there is no more any intersection. Now, taking derivatives and plugging the results would give us $D = -\dfrac{1}{4},\ E = 0,\ F = -3 \implies y_{p_2}(x) = -\dfrac{1}{4}x^4 - 3x^2$. Combining, we get, for arbitrary $c_{11}, c_{21}, c_{12}, c_{13} \in \mathbb{R}$,

$$\begin{aligned} y(x) &= y_{p_1}(x) + y_{p_2}(x) + y_h(x) \\ &= -\frac{1}{20}\sin(2x) - \frac{1}{4}x^4 - 3x^2 + c_{11} + c_{21}x + c_{12}e^x + c_{13}e^{-x} \end{aligned}$$

as the general solution.

Example: $y'' - 3y' - 2y = 2\sinh(x)$. Let us first determine the general homogeneous solution. Characteristic equation $\lambda^2 - 3\lambda + 2 = 0$ gives $\lambda_1 = 1,\ \lambda_2 = 2$ so that

$$y_h(x) = c_1e^x + c_2e^{2x}$$

Since $2\sinh(x) = e^x - e^{-x}$, we have the intersection "e^x" with the right hand side of the differential equation and $y_h(x)$. We thus form the modified candidate solution

$$y_p(x) = Axe^x + Be^{-x}$$

by multiplying e^x with x. Note that only the part due $f_1(x) = e^x$ in $2\sinh(x)$ need to be modified. Taking derivatives and plugging the results would give us $A = -1,\ B = -1\dfrac{1}{6}$ so that the general solution is

$$y(x) = y_p(x) + y_h(x) = -xe^x - \frac{1}{6}e^{-x} + c_1e^x + c_2e^{2x};\ c_1, c_2 \in \mathbb{R}$$

Remark: By the claim on page 27, the method of undetermined coefficients requires that the forcing function, i.e., the right hand side of the differential equation must be a linear combination of $\{1, \cos, \sin, \cosh, \sinh\}$ and their multiples by a power of x!!

1.6.2 Variation of Parameter Method

Given the general homogeneous solution to $L[y] = 0$ as

$$y_h(x) = c_1 y_1(x) + \cdots + c_n y_n(x)$$

for linearly independent $y_1, \ldots, y_n$ and constant $c_1, \ldots, c_n$ we form the candidate particular solution

$$y_p(x) = c_1(x) y_1(x) + \cdots + c_n(x) y_n(x)$$

for $L[y] = f(x)$, by varying the parameters $c_1, \ldots, c_n$

Second Order Case

$L[y] = y'' + p_1(x)y' + p_2(x) = f(x)$. Candidate solution is $y_p(x) = c_1(x)y_1(x) + c_2(x)y_2(x)$, where $L[y_1] = 0$, $L[y_2] = 0$ and $\{y_1, y_2\}$ is linearly independent. We have $y_j'' + p_1 y_j' + p_2 y_j = 0$ for $j = 1, 2$ and

$$\begin{aligned} y_p' &= c_1' y_1 + c_1 y_1' + c_2' y_2 + c_2 y_2' \\ y_p'' &= c_1'' y_1 + 2c_1' y_1' + c_1 y_1'' + c_2'' y_1 + 2c_2' y_2' + c_2 y_2'' \end{aligned}$$

so that $y'' + p_1 y' + p_2 y = f$ gives, for $y = y_p$

$$(c_1'' + c_1' p_1 + c_1 p_2) y_1 + (2c_1' + p_1 c_1) y_1' + y_1'' c_1 + (c_2'' + c_2' p_1 + c_2 p_2) y_2 + (2c_2' + p_1 c_2) y_2' + y_2'' c_2 = f$$

$$\implies \cancelto{0}{(y_1'' + p_1 y_1' + p_2 y_1)} c_1 + \cancelto{0}{(y_2'' + p_1 y_2' + p_2 y_2)} c_2 + p_1(c_1' y_1 + c_2' y_2) + 2(c_1' y_1' + c_2' y_2') + c_1'' y_1 + c_2'' y_2 = f$$

We now notice that $y_j'' + p_1 y_j' + p_2 y_j = 0$ for $j = 1, 2$ which gives the second line above. We now postulate that c_1, c_2 further satisfies

$$c_1' y_1 + c_2' y_2 = 0 \tag{1.28}$$

which implies $c_1'' y_1 + c_1' y_1' + c_2'' y_2 + c_2' y_2' = 0$ and results in

$$c_1' y_1' + c_2' y_2 = f \tag{1.29}$$

Combining (1.28) and (1.29), c_1', c_2' must be solution to

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} c_1' \\ c_2' \end{bmatrix} = \begin{bmatrix} 0 \\ f \end{bmatrix}$$

But $\det \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} = W[y_1, y_2] \neq 0$ so that the Cramer's Rule gives the solution

$$c_1 = \frac{\det \begin{bmatrix} 0 & f \\ y_1' & y_2' \end{bmatrix}}{\det \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}} = \frac{W_1}{W}, \; c_2 = \frac{\det \begin{bmatrix} y_1' & y_2' \\ 0 & f \end{bmatrix}}{\det \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}} = \frac{W_2}{W}$$

and

$$y_p(x) = y_1(x)\int^x \frac{W_1(\zeta)}{W(\zeta)}d\zeta + y_2(x)\int^x \frac{W_2(\zeta)}{W(\zeta)}d\zeta$$

<u>Example:</u> $y'' - 4y = 8e^{2x}$, $y_h(x) = c_1\underbrace{e^{2x}}_{y_1} + c_2\underbrace{e^{-2x}}_{y_2}$, $f = 8e^{2x}$

$$\begin{aligned}
\frac{W_1}{W} &= \frac{-fy_2}{y_1y_2''} = \frac{-8e^{2x}e^{-2x}}{-2e^{2x}e^{-2x} - 2e^{-2x}e^{2x}} = \\
\frac{W_2}{W} &= \frac{fy_1}{y_1y_2'' - y_1'y_2} = \frac{8e^{2x}e^{2x}}{-2e^{2x}e^{-2x} - 2e^{-2}} \\
y_p(x) &= e^{2x}\int^x 2d\zeta + e^{-2x}\int^x(-2e^{4\zeta})d\zeta \\
&= e^{2x}2x + e^{-2x}(-\frac{1}{2}e^{4x}) = 2xe^{2x} -
\end{aligned}$$

$y(x) = 2xe^{2x} - \dfrac{1}{2}e^{-2x} + Ae^{2x} + Be^{-2x} = 2xe^{2x} + \hat{A}e^{2x} + \hat{B}e^{-2x};\ \hat{A}, \hat{B} \in \mathbb{R}$

Higher Order Case

In n^{th} order case we have $c_1(x), \ldots, c_n(x)$ to be chosen suitably in the candidate particular solution

$$y_p(x) = c_1(x)y_1(x) + \cdots + c_n(x)y_n(x)$$

Consider the $n-1$ conditions

$$\left.\begin{array}{r} y_1c_1' + \cdots + y_nc_n' = 0 \\ y_1'c_1' + \cdots + y_n'c_n' = 0 \\ \vdots \\ y_1''c_1' + \cdots + y_n''c_n' = 0 \end{array}\right\} \begin{bmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{bmatrix} \begin{bmatrix} c_1' \\ c_2' \\ \vdots \\ c_n' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ f \end{bmatrix}$$

with the resulting condition

$$y_1^{(n-1)}c_1' + \cdots + y_n^{(n-1)} = 0$$

upon substitution into the differential equation $L[y_p] = f$.

<u>Example:</u> $x^3y''' + x^2y'' - 2xy' + 2y = \dfrac{2}{x};\ 0 < x < \infty$

$y_h(x)$: characteristic equation is $\lambda(\lambda-1)(\lambda-2) + \lambda(\lambda-1) - 2\lambda + 2 = \lambda^3 - 2\lambda^2 - \lambda + 2 = (\lambda-2)(\lambda^2-1) \implies \lambda_1 = 2, \lambda_2 = 1, \lambda_3 = -1$

$$y_h(x) = c_1\frac{1}{x} + c_2x + c_3x^2$$

In order to use the variation of parameter method, we first out the differential equation into standard form:

$$y''' + \frac{1}{x}y' - \frac{2}{x^2}y' + \frac{2}{x^3}y = \frac{2}{x^4}$$

so that $p_1 = x^{-1}, p_2 = -2x^{-2}, p_3 = 2x^{-3}, f = 2x^{-4}$. We have

$$W(x) = \det \begin{bmatrix} x^{-1} & x & x^2 \\ -x^2 & 1 & 2x \\ 2x^{-3} & 0 & 2 \end{bmatrix} = 2x^{-3}(2x^2 - x^2) + 2(x^{-1} + x^{-1}) = 6x^{-1}$$

$$W_1(x) = \det \begin{bmatrix} 0 & x & x^2 \\ 0 & 1 & 2x \\ 2x^{-4} & 0 & 2 \end{bmatrix} = 2x^{-4}(2x^2 - x^2) = 2x^{-2}$$

$$W_2(x) = \det \begin{bmatrix} x^{-1} & 0 & x^2 \\ -x^2 & 0 & 2x \\ 2x^{-3} & 2x^{-4} & 2 \end{bmatrix} = -2x^{-4}(2xx^{-1} - x^2x^{-2}) = -6x^{-4}$$

$$W_3(x) = \det \begin{bmatrix} x^{-1} & x & 0 \\ -x^2 & 1 & 0 \\ 2x^{-3} & 0 & 2x^{-4} \end{bmatrix} = 2x^{-4}(x^{-1} + xx^{-2}) = 4x^{-5}$$

Therefore,

$$\begin{aligned} y_p(x) &= x^{-1} \int^x \frac{1}{e\zeta} d\zeta + x \int^x (-\zeta^{-3}) d\zeta + x^2 \int^x \frac{2}{3}\zeta^{-4} d\zeta \\ &= \frac{1}{3}x^{-1}\ln(x) + \frac{1}{2}x^{-1} - \frac{2}{9}x^{-1} = \boxed{\frac{\ln(x)}{3x} - \frac{5}{18x}} \end{aligned}$$

Note that the $\left(-\frac{5}{18}x^{-1}\right)$ term can be dropped since $y_h(x)$ contains c_1x^{-1} term. Hence $\hat{y_p}(x) = \frac{\ln(x)}{3x}$ is also a particular solution.

1.6.3 Harmonic Oscillator

Equation of motion

$$mx'' + cx' + kx = f(t) = F_0\cos(\Omega t) \tag{1.30}$$

Homogeneous solution: $mx'' + cx' + kx = 0$ has the characteristic polynomial $m\lambda^2 + c\lambda + k = 0$ with roots

$$\lambda_1 = -\frac{c}{2m} + \sqrt{\frac{c^2 - 4mk}{4m^2}},\ \lambda_2 = -\frac{c}{2m} - \sqrt{\frac{c^2 - 4mk}{4m^2}}$$

1) **Underdamped Case:** $c^2 - 5mk < 0$ (non-real roots)

$$\alpha = \Re\lambda_1 = -\frac{c}{2m},\ \beta = \Im\lambda_1 = \sqrt{\frac{k}{m} - \frac{c^2}{4m^2}}$$

$$x_h(t) = e^{\alpha t}[A\cos(\beta t) + B\sin(\beta t)];\ A, B \in \mathbb{R}$$

2) **Critically Damped Case:** $c^2 - 4mk = 0$ (Repeated Roots)

$$x_h(t) = (A + Bt)e^{\alpha t};\ \alpha = -\frac{c}{2m}$$

3) **Overdamped Case:** $c^2 - 4mk > 0$ (Real distinct roots)

$$x_h(t) = e^{\alpha t}(Ae^{\gamma t} + Be^{-\gamma t});\ \gamma = \sqrt{\frac{c^2}{4m^2} - \frac{k}{m}}$$

Notes:

1) Non-real roots give oscillatory response even when $f(t) = 0$
2) All three cases give $\lim_{t\to\infty} x_h(t) = 0$
3) Small perturbations in k, m or c may cause a critically damped case turn into one of the other two cases.

Non-homogeneous Solutions: Let us use the method of undetermined coefficients and try

$$x_p(t) = C\cos(\Omega t) + D\sin(\Omega t)$$

Substitution into (1.30) results in

$$-mC\Omega^2\cos(\Omega t) - mD\Omega^2\sin(\Omega t) - cC\Omega\sin(\Omega t) + cD\cos(\Omega t) + k\cos(\Omega t) + kD\sin(\Omega t) = F_0\cos(\Omega t)$$
$$\implies \left.\begin{aligned} -mC\Omega^2 + cD\Omega + kC &= F_0 \\ -mD\Omega^2 - cC\Omega + kD &= 0 \end{aligned}\right\} C = \frac{(k - m\Omega^2)F_0}{\theta},\ D = \frac{c\Omega F_0}{\theta}$$

where $\theta = (k - m\Omega^2)^2 + c^2\Omega^2$ and hence,

$$x_p(t) = \frac{F_0}{\theta}[(k - m\Omega^2)\cos(\Omega t) + c\Omega\sin(\Omega t)]$$

This solution is valid if and only if $\{\cos(\Omega t), \sin(\Omega t)\}$ and $\{e^{\alpha t}\cos(\beta t), e^{\alpha t}\sin(\beta t)\}$ are non intersecting, which is the case if and only if $c \neq 0$ or $\Omega = (l - m\Omega^2)^2 + c^2\Omega^2 \neq 0$. In this critically damped and overdamped cases, there is no problem of intersection with homogeneous case basis functions. If, $c = 0$ and $\Omega = \sqrt{k/m}$, then the modified candidate solution $x_p(t) = Ct\cos(\Omega t) + Dt\sin(\Omega t)$ results in $C = 0$ and $D = \dfrac{F_0}{(2m\Omega)}$ so that $x_p(t) = \dfrac{F_0 t}{2m\Omega}\sin(\Omega t)$ is a function of ever growing oscillations. This situation has happened because the system is excited at its natural frequency $\sqrt{\dfrac{k}{m}}$ as $\Omega = \sqrt{\frac{k}{m}}$, and is called the phenomenon of response.

1.7 System of Linear Equations

Example

$$\begin{aligned} i_1 &= \dot{v}_1,\ i_2 = \dot{v}_2,\ i_3 = \dot{v}_3 \\ v_1 &= i_2 - i_1 + E(t) = \dot{v}_2 - \dot{v}_1 + E(t) \\ v_2 + i_2 - i_3 &= i_1 - i_2 \\ v_3 + i_3 &= i_2 - i_3 \\ &\implies \dot{v}_1 - \dot{v}_2 = E(t) - v_1,\ 2\dot{v}_2 - \dot{v}_1 - \dot{v}_3 = -v_2,\ 2\dot{v}_3 - \dot{v}_2 = -v_3 \end{aligned}$$

$$\Longrightarrow \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ - & -1 & 2 \end{bmatrix} \begin{bmatrix} \dot{v}_1 \\ \dot{v}_2 \\ \dot{v}_3 \end{bmatrix} = - \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} + \begin{bmatrix} E(t) \\ 0 \\ 0 \end{bmatrix}$$

$$\Longrightarrow \begin{bmatrix} \dot{v}_1 \\ \dot{v}_2 \\ \dot{v}_3 \end{bmatrix} = - \begin{bmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} + \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} E(t)$$

In general

$$\begin{aligned} \dot{x}_1 &= a_{11}(t)x_1 + a_{12}x_2 + \cdots + a_{1n}(t)x_n + f_1(t) \\ &\vdots \\ \dot{x}_n &= a_{n1}(t)x_1 + a_{n2}x_2 + \cdots + a_{nn}(t)x_n + f_n(t) \end{aligned} \tag{1.31}$$

where $a_{ij}(t)$ and $f_j(t)$ are given, x_j's to be found.

<u>Theorem:</u> If $f_1(t), \ldots, f_n(t), a_{jk}(t)$'s are continuous in a closed interval $I \subseteq \mathbb{R}$ containing a point $a \in I$ and if

$$x_1(a) = b_1, \ldots, x_n(a) = b_n \tag{1.32}$$

are given, then the system of equations (1.31) has a unique solution satisfying (1.32), valid on I.

<u>Note:</u> By solution of (1.31), we mean a set of continuous and differentiable functions $\{x_1(t, \ldots, x_n(t))\}$

<u>Remarks</u>

(1) Some higher order differential equations can be transformed into (1.31) by a suitable choice of x_j's

$$x''' + 2tx'' - x' + \sin(t)x = \cos(t)$$

Let

$$x_1 = x,\ x_2 = x',\ x_3 = x''$$

so that $\dot{x}_1 = x_2,\ \dot{x}_2 = x_3,\ \dot{x}_3 = -\sin(t)x_1 + x_2 - 2tx_3 + \cos(t)$. Which is a set of 3 equations in 3 unknowns $\{x_1, x_2, x_3\}$

(2) Any constant coefficient n^{th} order differential equation can be put into the for (1.31) with constant a_{jk}'s. Given

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1}y' + a_n y = f(t)$$

Let,

$$x_1 = y(t),\ x_2 = y'(t), \ldots,\ x_= y^{(n-1)}(t)$$

Then,

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_n & -a_{n-1} & a_{n-2} & \ldots & -a_1 \end{bmatrix} \begin{bmatrix} x_1 & \\ x_2 & \\ \vdots & x_{n-1} \\ x_n & \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ f \end{bmatrix}$$

Solution of (1.31) by Elimination: In the RC-circuit example on page 32, let $D = \dfrac{d}{dt}$. Then, the systems of differential equations becomes

$$\begin{bmatrix} D+3 & 2 & 1 \\ 2 & D+2 & 1 \\ 1 & 1 & D+1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} E(t)$$

Treating D as if it is a constant, we can solve for v_1, v_2, v_3 using e.g., Cramer's Rule. For instance

$$v_1 = \frac{\det \begin{bmatrix} 3E & 2 & 1 \\ 2E & D+2 & 1 \\ E & 1 & D+1 \end{bmatrix}}{\det \begin{bmatrix} D+3 & 2 & 1 \\ 2 & D+2 & 1 \\ 1 & 1 & D+1 \end{bmatrix}} = \frac{(3D^2+4D+1)E(t)}{D^3+6D^2+5D+1}$$

or, multiplying by the denominator,

$$(D^3+6D^2+5D+1)v_1(t) = f(t) := (3D^2+4D+1)E(t)$$

Note that this is a homogeneous, 3^{rd} order, linear, constant coefficient differential equation and can be solved by the techniques we have studied. This methid can be used even when (1.31) contains higher derivatives of $x_j(t)(j = 1, \ldots, n)$

Example:

$$\begin{aligned} m_1 x_1'' &= -k_1 x_1 - k_{12}(x_1 - x_2) + F_1 \\ m_2 x_2'' &= -k_2 x_2 + k_{12}(x_1 - x_2) + F_2 \end{aligned}$$

$$\begin{bmatrix} D^2+2 & -1 \\ -1 & D^2+2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$$

assuming $k_1 = k_{12} = k_2 = m_1 = m_2 = 1$. Solution:

$$(D^2+4D^2+3)x_1 = (D^2+2)F_1 + F_2 = F_1'' + 2F_1 + F_2$$

$$(D^2+4D^2+3)x_2 = (D^2)F_2 + F_1 = F_2'' + 2F_2 + F_1$$

Both x_1 and x_2 obey the same homogeneous differential equation with roots of the characteristic equation

$$\lambda_1 = i,\ \lambda_2 = -i,\ \lambda_3 = i\sqrt{3},\ \lambda_4 = -i\sqrt{3}$$

which gives

$$\begin{aligned} x_{1h}(t) &= A_1 \cos(t) + A_2 \sin(t) + A_3 \cos(\sqrt{3}t) + A_4 \sin(\sqrt{3}t) \\ x_{2h}(t) &= B_1 \cos(t) + B_2 \sin(t) + B_3 \cos(\sqrt{3}t) + B_4 \sin(\sqrt{3}t) \end{aligned}$$

for constants $A_j,\ B_j$. These 8 constants are not free. Because x_{1h} and x_{2h} are related by the original homogeneous equations

$$x_1'' = -2x_1 + x_2,\ x_2'' = -2x_2 + x_1$$

Substituting x_{1h} and x_{2h} into one of these, it is easy to get

$$A_1 = B_1,\ A_2 = B_2,\ A_3 = -B_3,\ A_4 = -B_4$$

The coefficients A_1, A_2, A_3, A_4 are then arbitrary. In fact, by theorem of existence and uniqueness on page 33, there should indeed be only 4 free parameter. Simply let $x_3 = x_1',\ x_4 = x_2'$ and apply the theorem.

Chapter 2

Difference Equations

A linear difference equation of order N is

$$a_0(n)y_{n+N} + a_1(n)y_{n+N-1} + \cdots + a_N(n)y_n = f(n),\ n \geq 0 \tag{2.1}$$

where $a_j(n), f(n)$ are specified and $y_n = y(n)$ is to be determined as a discrete function of n. When $f(n) \equiv 0$, (2.1) is called homogeneous and when $a_j(n)$ are constant, it is called constant coefficient.

Example: $y_{n+1} - \alpha y_n = \beta;\ \alpha, \beta \in \mathbb{R}$, is a 1^{st} order ($N = 1$), non-homogeneous, and constant coefficient difference equation. We note that

$$\begin{aligned} y_{n+1} &= \alpha y_n + \beta \\ &= \alpha^2(\alpha y_{n-1} + \beta) + \beta(\alpha + 1) \\ &\vdots \\ &= \alpha^n y_1 + \beta(\alpha^{n-1} + \alpha^{n-2} + \cdots + \alpha + 1) \\ &= \alpha^{n+1} y_0 + \beta(\alpha^n + \alpha^{n-1} + \cdots + \alpha + 1) \end{aligned}$$

By the identity

$$1 + \alpha + \cdots + \alpha^n = \frac{1 - \alpha^{n-1}}{1 - \alpha}, \alpha \neq 1$$

we have $y_{n+1} = \alpha^{n+1} y_0 + \beta(1 - \alpha^{n+1})/(1 - \alpha)$, or,

$$y_n = \begin{cases} \alpha^n y_0 + \beta \dfrac{1 - \alpha^n}{1 - \alpha} & , \alpha \neq 1 \\ y_0 + \beta n & , \alpha = 1 \end{cases}$$

Observe that $y_n|_{\beta=0} = \alpha^n y_0$ is the homogeneous solutions and $y_n|_{y_0=0} = \beta \dfrac{1-\alpha^n}{1-\alpha}$ is a particular solution when $\alpha \neq 1$, or $y_n|_{y_0=0}\, \beta n$ when $\alpha = 1$. Consider (2.1) with $a_j(n) = a_j \in \mathbb{R}$, constant coefficient. Let us consider the general homogeneous case, $f_n(0) \equiv 0$, of (2.1). Trying a candidate $y_n = \rho^n;\ \rho \in \mathbb{R}$, it is easy to see that, ρ is a root of the characteristic equation.

$$\rho^N + a_1 \rho^{N-1} + \cdots + a_{N-1}\rho + a_N = 0 \tag{2.2}$$

Conversely, any root $\rho \in \mathbb{C}$ of thus equation gives a homogeneous solution $y_n = \rho^n$. It is also not difficult to see that if rho is a multiple root, then new linearly independent solutions can be generated by multiplication with a power of $n : n, n^2, n^3, \ldots$. It is then easy to arrive at: If $\rho_1, \ldots, \rho_k$ are the k distinct roots of (2.2) with multiplicity $m_1, \ldots, m_k$, then the general solution of (2.1) with $f(n) \equiv 0$ and $a_j(n) = a_j \in \mathbb{R}$ is given by

$$y_n = \sum_{j=1}^{k} [c_{1j} + nc_{2j} + \cdots + n^{m_j - 1} c_{m_j j} \rho) j^n]$$

where $c_{kj} \in \mathbb{C}$ and c_{kj}'s occur in conjugate pairs if and only if ρ_j is non-real. If $y_0, \ldots, y_{n-1}$ are specified, then c_{kj}'s are uniquely found.

Example: $y_{n+4} - 2y_{n+2} + y_n = 1$, Let us write $y_n = h_n + p_n$, where h_n is the general homogeneous solution and p_n is a particular solution. The characteristic equation

$$\rho^4 - 2\rho^2 + \rho = (\rho^2 - \rho + 1)(\rho^2 + 2\rho + 1) = (\rho - 1)^2 (\rho + 1)^2 = 0$$

has roots $\rho_1 = 1,\ \rho_2 = -1$ with multiplicities $m_1 = 2,\ m_2 = 2$ so that

$$h_n = (c_{11} + nc_{21}) + (c_{21} + nc_{22})(-1)^n;\ n \geq 0 \tag{2.3}$$

for arbitrary $c_{kj} \in \mathbb{R}$. A particular solution p_n can be determined by the method of undetermined coefficients. A modified candidate solution is in the form $p_m = An^2$ (modified from $p_n = A$ since $\{1, n\}$ are among functions of h_n). Then,

$$h_{n+4} - 2h_{n+2} + h_n = A(n+4)^2 - 2A(n+2)^2 + An^2 = 8A = 1 \implies A = \frac{1}{8} \implies p_n = \frac{1}{8}n^2$$

It follows that $y_n = h_n + \frac{1}{8}n^2$ with h_n given by (2.3) is the general solution.

Example: $y_{n+1} - 3y_n = \sin(2n)$. We have $h_n = A3^n;\ A \in \mathbb{R}$, as the general homogeneous solution. The candidate for p_n is

$$p_n = B\cos(2n) + C\sin(2n)$$

Then, substituting into $p_{n+1} - 3p_n = \sin(2n)$, we get

$$\begin{aligned}
&B\cos(2n+2) + C\sin(2n+2) - 3B\cos(2n) - 3C\sin(2n) = \sin(2n) \\
&\implies B[\cos(2n)\cos(2) - \sin(2n)\sin(2)] + C[\sin(2n)\cos(2) + \cos(2n)\sin(2)] - 3B\cos(2n) \\
&- 3C\sin(2n) = \sin(2n) \\
&\implies \begin{bmatrix} -\sin(2) & \cos(2) - 3 \\ \cos(2) - 3 & \sin(2) \end{bmatrix} \begin{bmatrix} B \\ C \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \implies C = \frac{\cos(2) - 3}{10 - 6\cos(2)},\ B = \frac{-\sin(2)}{10 - 6\cos(2)}
\end{aligned}$$

It follows that

$$p_n = \frac{1}{10 - 6\cos(2n)}[(\cos(2) - 3)\sin(2n) - \sin(2)\cos(2n)]$$

Example: (Fibonacci Equation) A tree gives a branch every other year. The figure shows that the number of branches in each year follows the sequence

$$1, 1, 2, 3, 5, 8, 13, \ldots$$

The number of branches in year $n+2$ is the sum of those in the previous two years $n+1,\ n$:

$$y_{n+2} = y_{n+1} + y_n,\ y_1 = 1,\ y_2 = 1$$

If we let $f_n = y_{n+1}$, then

$$f_{n+2} = f_{n+1} + f_n;\ y_1 = f_0 = 1,\ y_2 = f_1 = 1$$

which has the characteristic equation $\rho^2 - \rho - 1 = 0$ with roots $\rho_{1,2} = \dfrac{1 \pm \sqrt{5}}{2}$, so that the general solution is with $c_1, c_2 \in \mathbb{R}$,

$$f_n = c_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + c_2 \left(\frac{1-\sqrt{5}}{2}\right)^n,\ n \geq 0$$

The initial values $f_0 = f_1 = 1$ gives $c_1 + c_2 = 1,\ c_1 - c_2 = \dfrac{1}{\sqrt{5}}$, so that

$$f_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right],\ n \geq 0$$

$$y_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n} - \left(\frac{1-\sqrt{5}}{2}\right)^{n}\right],\ n \geq 1$$

$$G = \lim_{n\to\infty} \frac{y_{n+1}}{y_n} = \frac{1+\sqrt{5}}{2} \text{ is called the Golden Ratio}$$

Chapter 3

Numerical Solutions

Let us consider the initial value problem on $I = [a, A] \in \mathbb{R}$

$$y' = f(x, y),\ y(a) = b$$

where f is a general (perhaps non-linear) function of x and y.

3.1 Euler's Method

Partition the interval $[a, A]$ into

$$a = x_0,\ x_1 = a + h,\ \ldots, x_k = a + kh$$

and apply the algorithm

$$y_{n+1} = y_n = f(x_n, y_n)h;\ n = 0, 1, 2, \ldots$$

which obtains $y_1, y_2, \ldots$ based in the information of "direction field".

Example: $y' = y + 2x - x^2,\ y(0) = 1,\ 0 \leq x < \infty$The exact solution is $y(x) = e^x + x^2$, obtained from the 1^{st} order differential equation solution with $a = 0,\ b = 1,\ p(x) = -1,\ q(x) = 2x - x^2$. Let us choose the step size $h = 0.5$ so that

$$\begin{aligned}
y_1 &= y_0 + (y_0 + 2x_0 - x_0^2)h = 1 + (1 - 0 - 0)0.5 = 1.5 \\
y_2 &= y_1 + (y_1 + 2x_1 - x_1^2)h = 1.5 + (1.5 + 1 - 0.25)0.5 = 2.625 \\
y_3 &= y_3 + (y_2 + 2x_2 - x_0^2)h = 2.625 + (2.625 + 2 - 1)0.5 = 4.4375
\end{aligned}$$

Since $y(x_1) = 1.8987,\ y(x_2) = 3.7183,\ y(x_3) = 6.7317$ is obtained from the exact solution, the error becomes increasingly and quickly large between y_k and $y(x_k)$. Of course, this is because h is quite large and we expect that the error $e_k = y(x_k) - y_k$ will decrease and go to zero as $h \to 0$. This is an issue of convergence.

3.1.1 Convergence of Euler's Algorithm

The error at step n is

$$e_n = y(x_n) - y_n = y(x_n) - y_{n-1} - f(x_{n-1}, y_{n-1})h$$

By Taylor's formula, we have

$$y(x_n) = y_(x_{n-1}) + y'(x_{n-1})h + \frac{y''(\zeta)}{2}h^2$$

for some $\zeta \in [x_{n-1}, x_n]$. Hence,

$$e_n = y(x_{n-1}) - y_{n-1} + [y'(x_{n-1}) - f(x_{n-1}, y_{n-1})h + \frac{y''(\zeta)}{2}h^2]$$

If we assume that no error was made at step $n-1$, then $y_{n-1} = y(x_{n-1})$ so that, we also have, $f(x_{n-1}, y(x_{n-1})) = f(x_{n-1}, y_{n-1})$ with $y'(x_{n-1}) = f(x_{n-1}, y(x_{n-1}))$ from the differential equation. Therefore

$$e_n = \frac{y''(\zeta)}{2}h^2 \text{ (single step error)}$$

varies with h^2. This is denoted by the term order of h^2 and by the notation

$$e_n = Q(h^2)$$

This error accumulates

$$\begin{aligned} E_n &\cong e_1 + \cdots + e_n \\ &= nQ(h^2) \\ &= Q(h^2)\frac{nh}{h} \\ &= Q(h^2)\frac{x_n - x_0}{h} = Q(h) \end{aligned}$$

i.e., the cumulative error E_n grows with $Q(h)$. The Euler algorithm is thus said to be a 1^{st} order algorithm as it converges with a speed measured with h.

Note: E_n is only approximately equal to

$$e_1 + \cdots + e_n$$

since the slopes of lines $L_1 and L_2$ shown in the figure are different:

$$\left.\begin{array}{ll} \text{slope of } L_2: & f(x_1, y_1) \\ \text{slope of } L_1: & f(x_1, y(x_1)) \end{array}\right\} \beta \cong e_1$$

Because $u_1 \neq y(x_1)$, we have $\beta \neq e_1$ but $\beta \cong e_1$

3.2 Midpoint Rule

In Euler;s method, we used the approximations

$$y'(x_1) \cong f(x_n, y_n),\ y'(x_n) = \frac{y(x_{n+1} - y(x_n))}{h}$$

This is, geometrically, approximation the slope of $y'(x_n)$ by the slope of the line BC. Instead, we can approximate it with the slope of the line AC

$$y'(x_n) \cong \frac{y(x_{n+1}) - y(x_{n-1})}{2h}$$

and use $y'(x_n) \cong f(x_n, y_n)$ to obtain a different algorithm.

$$y_{n+1} = y_{n-1} + f(x_n, y_n)2h;\ n = 1, 2, \ldots \tag{3.1}$$

This is called the midpoint rule and uses a different approximation than Euler's. Of course, to start the algorithm (3.1), we need initial values y_0, y_1. The usual practice is to get y_1 from y_0 by Euler's method, i.e., by $y_1 = y_0 + f(x_0, y_0)h$

3.2.1 Convergence of Midpoint Rule

Let is write, by Taylor's formula

$$y(x_{n+1}) = y(x_n) + y'(x_n)h + \frac{y''(x_n)}{2}h^2 + \frac{y'''(\eta)}{6}h^3$$

for some $h > 0$. Since $h = x_n - x_{n-1}$

$$y(x_{n-1}) = y(x_n) - y'(x_n)h + \frac{y''(x_n)}{2}h^2 - \frac{y'''(\zeta)}{6}h^3$$

Here, η, ζ are some points in the interval (x_{n-1}, x_{n+1}). Subtracting the two, we have

$$y(x_{n+1}) - y(x_{n-1}) = y'(x_n)2h + \frac{y'''(\eta) + y'''(\zeta)}{6}h^3$$

so that the two step error can be computed by, using $y_{n+1} - y_{n-1} = f(x_n, y_n)2h$ and noting that

$$e_{n+1} - e_{n-1} = [y'(x_n) - f(x_n, y_n)]2h + \frac{y'''(\eta) + y'''(\zeta)}{6}h^3$$

This gives $e_{n+1} - e_{n-1} = Q(h^3)$ provided $y'(x_n) = f(x_n, y(x_n)) = f(x_n, y_n)$. It follows that e_{n+1} grows with $Q(h^3)$. The cumulative error grows with $Q(h^2)$ so that the midpoint rule (3.1) is 2^{nd} order algorithm. The convergence of the midpoint rule is faster than of Euler's method. These algorithms are easily adapted to systems of differential equations. Consider

$$u' = f(x, u, v),\ u(a) = u_0$$
$$v' = g(x, u, v),\ v(a) = v_0$$

Euler's method give

$$u_{n+1} = u_n + f(x_b, y_n, v_n)h$$
$$v_{n+1} = v_n + f(x_b, y_n, v_n)h$$

Example: $u' = x + v,\ v' = uv^2;\ u(0) = 0,\ v(0) = 1$. Let us choose $h = 0.1$, then

$$\underline{\text{n=0:}}\ u_1 = u_0 + (x_0 + v_0)h = 0 + (0+1)0.1 = 0.1$$
$$v_1 = v_0 + u_0 v_0^2 h = 1 + 0(1)^2 0.1 = 1$$
$$\underline{\text{n=1:}}\ u_2 = u_1 + (x_1 + v_1)h = 0.1 + (0.1+1)0.1 = 0.21$$
$$v_2 = v_1 + u_1 v_1^2 h = 1 + 0.1(1)^2 0.1 = 1.01$$

3.3 Stability of Numerical Algorithms

An algorithm gives incorrect results for two reasons:

i) Sensitivity to initial conditions.

ii) Instability of the algorithm

3.3.1 Sensitivity

The differential equation $y' - 2y = -6e^{-4x}$ has the general solution

$$y = e^{-4x} + Ce^{2x} = \begin{cases} e^{-4x} & ,\ y(0) = 1\ (C = 0) \\ Ce^{2x} + e^{-4x} & ,\ y(0) \neq 1\ (C \neq 0) \end{cases}$$

Suppose we apply the midpoint rule with initial value $y(0) = 1$. Even at the first step, y_1 will be at a direction field, which generates a solution for another initial condition corresponding to $C \neq 0$. The algorithm will give values such that $y_n \to \infty$ as $n \to \infty$.

3.3.2 Instability

Round-off errors in the machine accumulate and give incorrect results after a certain step. Such algorithms are called unstable.

Example: $y' = -2y,\ y(0) = 1$. Apply Midpoint Rule: $y_{n+1} = y_{n-1} + (-2y_n)2h,\ h = 0.05$. After $x = 2$, the rule starts to give oscillations about $y = 0$ with ever-growing oscillations. Suppose e_0 has a round-off error of ϵ, i.e., $e_0 = \epsilon$, and calculate how much the perturbed solution deviates from the exact solution at $x = x_n = nh$. The differential equation to which Midpoint Rule is applied is more general than that of the example

$$y' = Ay,\ y(0) = 1,\ A \in \mathbb{R}$$

$$y_{n+1} = y_{n-1} + 2Ay_n h,\ y_0 = 1,\ n = 0, 1, 2, \ldots \tag{3.2}$$

Since $e_0 = \epsilon$, we have $y_0^* = y(0) - \epsilon = 1 - \epsilon$ and the rule generates

$$y_{n+1}^* = y_{n-1}^* + 2Ahy_n^*,\ y_0^* = 1 - \epsilon,\ n = 1, 2, 3, \ldots \tag{3.3}$$

The error also obeys, by (3.2) and (3.3), a similar equation and its roots are

$$\rho^2 - 2Ah\rho - 1 = 0,\ \rho_{1,2} = A \pm \sqrt{1 + A^2h^2}$$

and the general solution is

$$e_n = c_1(Ah + \sqrt{1 + A^2h^2})^n + c_2(Ah - \sqrt{1 + A^2h^2})^n,\ n \geq 0$$

Assuming h is small, we have $1 + A^2h^2 \sim 1$ so that

$$\begin{aligned} e_n &\cong c_1(Ah + 1)^n + c_2(-1)^n(1 - Ah)^n \\ &\cong c_1e^{n\ln(1+Ah)} + c_2(-1)^n e^{n\ln(1-Ah)},\ (\ln(1 + x) \cong x^2 - \frac{x^2}{2} + \dots) \\ &\cong c_1e^{Anh} + c_2(-1)^n e^{-Anh},\ (x_1 = h,\ x_n = nh) \\ &\cong c_1e^{Ax_n} + c_2(-1)^n e^{-Ax_n} \end{aligned}$$

We now assume $e_1 = 0$ and also use $e_0 = \epsilon$ to get

$$\begin{aligned} &c_1 + c_2 = \epsilon,\ c_1e^{Ax_1} - c_2e^{-Ax_1} = 0 \\ \implies &e_n \cong \frac{\epsilon}{e^{Ax_1} + e^{-Ax_1}}[e^{A(x_n-x_1)} + (-1)^n e^{-A(x_n-x_1)}] \end{aligned}$$

<u>$A > 0$</u>**:** The exact solution is the growing exponentials $y(x_n) = e^{Ax_n}$. For small $|\epsilon|$, we have

$$e_n \xrightarrow[n\to\infty]{} \frac{\epsilon e^{A(x_n-x_1)}}{e^{Ax_1} + e^{-Ax_1}} = \frac{\epsilon e^{-Ax_1}}{e^{Ax_1} + e^{-Ax_1}}e^{Ax_n} << e^{Ax_n}$$

so that, for large n, error due to round-off is much smaller than the exact value and the deviation is not noticeable.The rule is stable for $A > 0$.

<u>$A < 0$</u>**:** The exact solution is the diminishing exponential $y(x_n) = e^{Ax_n}$ and for small ϵ

$$e_n \xrightarrow[n\to\infty]{} \frac{\epsilon(-1)^n e^{-A(x_n-x_1)}}{e^{Ax_1} + e^{-Ax_1}} = \frac{\epsilon e^{Ax_1}}{e^{Ax_1} + e^{-Ax_1}}(-1)^n e^{-Ax_n}$$

so that the round-off error, for large n, becomes more noticeable and grows with oscillatory behaviour. The Midpoint Rule is unstable for $A < 0$

<u>**Note:**</u> Technique for examining stability extends to differential equations of the type $y' = f(x, y)$ by linearisation about $y = 0$:

$$y' \cong f(x, 0) + \left.\frac{\partial f}{\partial y}\right|_{y=0} y = \left.\frac{\partial f}{\partial x}\right|_{y=0} y$$

provided $y = 0$ is an equilibrium point of $y' = f(x, y)$

Chapter 4

Functions of Several Variables

The set of n-tuples is

$$\mathbb{R}^n = \{\underline{x} = (x_1, \ldots, x_n),\ x_j \in \mathbb{R}\}$$

and a function of n-variables is

$$f : \mathbb{R}^n \to \mathbb{R} : x \mapsto f(x) : f(x_1, \ldots, x_n)$$

The distance of x to a point x' is $d(\underline{x}, \underline{x}') := \sqrt{(x_1' - x_1)^2 + \cdots + (x_n' - x_n)^2}$, where $\underline{x} = (x_1, \ldots, x_n)$ and $\underline{x}' \equiv (x_1', \ldots, x_n')$. Note that

$$\begin{aligned} n = 1 &: \ d(\underline{x}, \underline{x}') = |x_1' - x_1| \\ n = 2 &: \ d(\underline{x}, \underline{x}') = \sqrt{(x_1' - x_1)^2 + (x_2' - x_2)^2} \\ n = 3 &: \ d(\underline{x}, \underline{x}') = \sqrt{\sum_{j=1}^{3} (x_j - x_j')^2} \end{aligned}$$

The neighbour of $\underline{x}'$ is

$$N(\underline{x}', r) = \{x \in \mathbb{R}^n : \ d(\underline{x}, \underline{x}') < r\}$$

$$\begin{aligned} n = 1 &: \ \text{an interval} \\ n = 2 &: \ \text{disk of radius} \\ n = 3 &: \ \text{sphere of radius} \end{aligned}$$

A set $S \subseteq \mathbb{R}^n$ is connected if each pair of points in S can be joined by a finite number of line segments connected end-to-end. Note that a straight line in n-dimensional space has the parametric equation

$$x_1 = a_1 + b_1 t, \ldots, x_n = a_n + b_n t;\ a_j, b_j \in \mathbb{R},\ t \in \mathbb{R} \text{ or } [\alpha, \beta]$$

A point in $\underline{x} \in \mathbb{R}^n$ is an interior point of S if

$$\underline{x} \in S \,\&\, N(\underline{x}, r) \subseteq S \text{ for some } r > 0$$

It is a boundary point of S if $N(\underline{x}, r)$, for any $r > 0$, contains points from S as well as outside S. A connected set is a domain or an open set if it contains none of its boundary points. It is a closed set of it contains all of its boundary points. It is easy to see that a connected set S is an open set of and only if every point of S is an interior point.

Example

$N(\underline{x}', r)$ is an open set for every $r > 0$. $N(\underline{x}', r)$ together with its boundary (circle in $n = 2$, sphere-shell in $n = 3$, etc.) is called a closed set.

Example

Note that $\mathbb{R}$ is both open and closed, since it has no boundary points. The set in the first example on this page is neither open nor closed.

4.1 Limit of a Function

The limit of $f(\underline{x})$ as $\underline{x}$ approaches $\underline{x}'$ is L, $\lim_{\underline{x}\to\underline{x}'} f(x) = L$, if given any $\epsilon > 0$, there is $\delta > 0$ such that

$$d(\underline{x}, \underline{x}') < \delta \implies |f(\underline{x} - L) < \epsilon|$$

for every $\underline{x} \in N(\underline{x}', \delta)$ in the domain of definition of f. Thus, no matter how small $\epsilon > 0$ is give, there is a f-neighbourhood of $\underline{x}'$ such that $f(\underline{x})$ is ϵ-close to L for every $\underline{x}$ in that neighbourhood. $f(\underline{x})$ is continuous at $\underline{x}'$ if

$$\lim_{\underline{x}\to\underline{x}'} f(\underline{x}) = f(\underline{x}')$$

Example

- $H(x) = \begin{cases} 1, & x > 0 \\ 0, & x < 0 \end{cases}$ and $H(0) = \dfrac{1}{2}$ is continuous everywhere except at $x = 0$, where the limit $\lim_{x\to 0} H(x)$ does not exist.
- $f(\underline{x}) = \dfrac{1}{x_1^2 + x_2^2 + x_3^2}$ is continuous everywhere except at $\underline{x} = \underline{0}$ because neither $\lim_{\underline{x}\to\underline{0}} f(\underline{x})$ nor $f(\underline{0})$ exists.
- $f(x_1, x_2) = x_1^2 + e^{x_2}\sin(x_1 x_2^2)$ is continuous in the whole x_1x_2-plane since it is the product, sum, composite of continuous functions.

4.2 Partial Derivatives (n=2)

Suppose the domain of definition of $f(x, y)$ includes a neighbourhood of (x_0, y_0). Then,

$$f_x(x_0, y_0) = \left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} := \lim_{\Delta x\to 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x}$$

$$f_y(x_0, y_0) = \left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} := \lim_{\Delta y\to 0} \frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta y}$$

are the partial derivatives of f with respect to x and y at (x_0, y_0). If f_x and f_y exist in a neighbourhood of (x_0, y_0), then they are functions of (x, y) and one can define second order derivatives

$$f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2},\ f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$$

$$f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x},\ f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}$$

Fact: If $f_x,\ f_y,\ f_{xy},\ f_{yx},$ are all continuous in a neighbourhood of (x_0, y_0), then $f_{xy} = f_{yx}$ at (x_0, y_0)

Remark: Unlike functions of one variable, $f_x,\ f_y$ may exist at (x_0, y_0) without f being continuous at (x_0, y_0).

Example

$$f(x, y) = \begin{cases} 1,\ x = 0 \text{ or } y = 0 \\ 0,\ x \neq 0 \text{ and } y \neq 0 \end{cases}$$

$$f_x(0,0) = \lim_{\Delta x \to 0} \frac{f(\Delta x, 0) - f(0,0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{1-1}{\Delta x} = 0$$

$$f_y(0,0) = \lim_{\Delta y \to 0} \frac{f(0, \Delta y) - f(0,0)}{\Delta y} = \lim_{\Delta y \to 0} \frac{1-1}{\Delta y} = 0$$

but $\lim\limits_{\underline{x} \to \underline{0}} f(x, y)$ does not exist since it depends on how we approach the origin. It follows that f is not continuous at $(0, 0)$.

Example

Consider $f(x_1, \ldots, x_n)$ such that

$$f_{x_1 x_1} + \cdots + f_{x_n x_n} = 0 \tag{4.1}$$

Let us try to find a solution. Postulate $f = (x_1^2 + \cdots + x_n^2)^\alpha$ for some $\alpha \in \mathbb{R}$. Then

$$\begin{aligned} f_{x_k x_k} &= \frac{\partial}{\partial x_k}[2\alpha x_k (x_1^2 + \cdots + x_n^2)^{\alpha - 1}] \\ &= 2\alpha(x_1^2 + \cdots + x_n^2)^{\alpha-1} + 4\alpha(\alpha - 1)x_k^2(x_1^2 + \cdots + x_n^2)^{\alpha-2} \\ &= 2\alpha(x_1^2 + \cdots + x_n^2)^{\alpha-2}[x_1^2 + \cdots + x_n^2 + 2(\alpha-1)x_k^2] \end{aligned} \tag{4.2}$$

Now, (4.1) holds if and only if

$$\sum_{k=1}^{n}[x_1^2 + \cdots + x_n^2 + 2(\alpha - 1)x_k^2] = [n + 2(\alpha - 1)](x_1^2 + \cdots + x_n^2) = 0,$$

which holds if and only if $a = \frac{2-n}{2}$ and $f = (x_1^2 + \cdots + x_n^2)^{\frac{2-n}{2}}$.

4.3 Composite Functions and Chain Differentiation

Let us first consider a function of one variable $f(x)$, which is differentiable over $\mathbb{X} = \{x : a < x < b\}$. Suppose $x = x(t)$ is in turn differentiable over $T = \{t : \alpha < t < \beta$, such that $x(t) \in \mathbb{X}$ $\forall t \in T$. Then, $F(t) := f(x(t))$ is defined over T and is differentiable over T with

$$\frac{dF}{dt} = \frac{df}{dx} \cdot \frac{dx}{dx}$$

F is called the "composite function of t" or the composition of f and x. This 1-D result can be generalized.

Theorem: Let $f(x, y)$, $f_x(x, y)$ and $f_y(x, y)$ be continuous over an open region (set) R in xy-plane. Let $x = x(t)$, $y = y(t)$ be differentiable over an interval $T \subseteq \mathbb{R}$ such that $(x(t), y(t)) \in \mathbb{R}$. Then $F(t) := f(x(t), y(t))$ is a differentiable function of t over T and

$$\frac{dF}{dT} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \tag{4.3}$$

Proof: Let Δt be small enough so that t and $t + \Delta t$ are both in T. Let

$$\Delta x := x(t + \Delta t) - x(t),\ \Delta y := y(t + \Delta t) - y(t)$$

Then,

$$\begin{aligned} \Delta F : &= F(t + \Delta t) - F(t) = f(x(t + \Delta t), y(t + \Delta t)) - f(x(t), y(t)) \\ &= f(x + \Delta x, y + \Delta y) - f(x, y) \\ &= f(x + \Delta x, y + \Delta y) - f(x, y + \Delta y) + f(x, y + \Delta y) - f(x, y) \\ &= f(S) - f(Q) + f(Q) - f(P) \end{aligned}$$

By mean-value theorem, the points $\hat{P}$, $\hat{Q}$ as shown above, subject to,

$$f(Q) - f(P) = f_y(\hat{P})\Delta y,\ f(S) - f(Q) = f_x(\hat{Q})\Delta x$$

Hence

$$\frac{\Delta F}{\Delta t} = \frac{F(t + \Delta t) - F(t)}{\Delta t} = f_y(\hat{P})\frac{\Delta y}{\Delta t} + f_x(\hat{Q})\frac{\Delta x}{\Delta t}$$

Taking limits as $\Delta t \to 0$ and noting that $\Delta x \to 0$, $\Delta y \to 0$, $\hat{Q} \to P$, $\hat{P} \to P$ as $\Delta t \to 0$, we obtain $\frac{dF}{dt} = f_x(P)\frac{dx}{dt} + f_y(P)\frac{dy}{dt}$.

Example

$r(x, y) = x^2 y - e^{2y}$, $x = 3t^2$, $y = \sin t$; $1 < t < 4$. Now, $R(t) = r(x(t), y(t)) = 9t^4 \sin t - e^{2 \sin t}$. By (4.3), we get

$$\begin{aligned} \dot{R}(t) &= 2xy \cdot 6t + (x^2 - 2e^{2y}) \cos t \\ &= 36t^3 \sin t + (9t^4 - 2e^{2 \sin t}) \cos t \end{aligned}$$

, which coincides with $\dot{R(t)}$ obtained by direct derivative of $9t^4 - 2e^{2sint}$ with respect to t.

4.4 Taylor's Formula in 1D and 2D

Recall that in one dimensional case, by Fundamental Theorem of Calculus, we can write for $x, a \in \mathbb{R}$,

$$f(x) = f(a) + \int_a^x \dot{f}(x)dx, \; \dot{f}(x) = \dot{f}(a) + \int_a^x \ddot{f}(x)dx,$$

which give

$$\begin{aligned} f(x) &= f(a) + \int_a^x \left[\dot{f}(x) + \ddot{f}(x)dx\right] dx \\ &= f(a) + \dot{f}(a)(x-a) + \int_a^x \int_a^x \ddot{f}(x)dxdx \end{aligned}$$

that generalizes to

$$f(x) = \sum_{i=0}^{n-1} f^{(i)}(a) \cdot \frac{(x-a)^i}{i!} + R_n(x) \tag{4.4}$$

where

$$R_n(x) = \int_a^x \dots \int_a^x f^{(n)}(x)dx \dots dx \tag{4.5}$$

provided f is differentiable n-time in the interval a, x. We now show that,

$$R_n(x) = \frac{f^{(n)}(\zeta)}{n!}(x-a)^n \tag{4.6}$$

for some ζ between a and x. To see this let m and M be the minimum and maximum of $f^{(n)}(z)$ for z in between a and x, which both exist provided $f^{(n)}$ is continuous in between a and x. Then, (4.5) gives

$$m \int_a^x \dots \int_a^x dx \dots dx \le R_n(x) \le M \int_a^x \dots \int_a^x dx \dots dx$$

or

$$m\frac{(x-a)^n}{n!} \le R_n(x) \le M\frac{(x-a)^n}{n!}$$

Now since $f^{(n)}(z)$ must assume all values between m and M by its continuity, it follows that for some ζ in between a and x,

$$R_n(x) = \frac{f^{(n)}(\zeta)}{n!}(x-a)^n$$

as claimed (4.6). The expression in (4.6) is called the Lagrange form of remainder for $R_n(x)$. The expansion (4.4) is called the Taylor's formula with n-th order error $R_n(x)$. In fact, if one approximates

$$f(x) \cong \sum_{i=0}^{n-1} f^{(i)}(a) \cdot \frac{(x-a)^i}{i!}$$

then the "error of approximation" is given by $|R_n(x)|$

Example
Approximate e^{-x} over $0.7 < x < 1.3$ by a second order polynomial of x : let us choose $a = 1$ so that (4.4) with $n = 3$ gives

$$f(x) = e^{-x} = e^{-1} - e^{-1}(x-1) + e^{-1}\frac{(x-1)^2}{2} + R_3(x) \tag{4.7}$$

with $R_3(x) = -\frac{e^{-\zeta}}{3!}(x-1)^3$, $\zeta \in [1, x]$ or $\zeta \in [x, 1]$, whichever inequality $x \geq 1$ or $1 \geq x$ holds. Note that

$$|R_3(x)| = e^{-\zeta}\frac{|x-1|^3}{6} < e^{-0.7}\frac{(0.3)^3}{6} \cong 0.0022 =: err,$$

the error of approximation made in (4.7) is less than 'err' as long as $0.7 < x < 1.3$.

4.5 Taylor Series

If derivatives of all orders of f exist and if $\lim_{n\to\infty} |R_n(x)| = 0$ for all x in an interval $\mathbb{I}$ including a, then

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

is the Taylor Series of f about a. It is convergent for all $x \in \mathbb{I}$

Example
With $a = 0$,

$$e^{-x} = \sum_{k=0}^{\infty}(-1)^k \frac{x^k}{k!}$$

Now , $R_n(x) = f^{(n)}(x)\frac{(x-a)^n}{n!} = (-1)^n \frac{e^{-\zeta}}{n!}x^n$ and

$$\lim_{n\to\infty} |R_n(x)| = \lim_{n\to\infty} \frac{e^{-\zeta}}{n!}|x|^n = 0$$

for all ζ between 0 and x and for all $x \in \mathbb{R}$, it follows that Taylor Series of e^{-x} exists for all x about 0. ($\lim_{n\to\infty} \frac{|x|^n}{n!} = 0$ for any $x \in \mathbb{R}$, which can be seen by "squeezing the tail".)

We now generalize (4.4) and (4.6) to find 2D and then to any dimension. Given $f(x, y)$, let R be and open set in the xy-plane and consider a line extending from (a, b) to (x_0, y_0). We can parametrize any point (x, y) on the line by

$$x = a + (x_0 - a)t, \; y = b + (y_0 - a)t,$$

where $t \in [0, 1]$. The composite function $F(t)$ given by $F(t) = f(x(t), y(t))$ has the expression

$$F(t) = f(a + (x_0 - a)t, b + (y_0 - b)t)$$

and it can be expanded about $t = 0$ using n^{th} order Taylor's Formula as

$$F(t) = F(0) + \dot{F}(0) + \cdots + \frac{F^{(n-1)}(0)}{(n-1)!}t^{n-1} + R_n(t) \tag{4.8}$$

where

$$R_n(t) = \frac{F^{(n)}(\tau)}{n!}t^n \text{ for some } \tau \in (0,1)$$

provided n^{th} order derivative of F exists and is continuous. By Chain Differentiation

$$F'(t) = f_x\frac{dx}{dt} + f_y\frac{dy}{dt} = f_x|_{(a,b)}(x_0 - a) + f_y|_{(a,b)}(y_0 - b)$$

$$\begin{aligned} F''(t) &= \left(f_{xx}\frac{dx}{dt} + f_{xy}\frac{dy}{dt}\right)\frac{dx}{dt} + \left(f_{yy}\frac{dy}{dt} + f_{yx}\frac{dx}{dt}\right)\frac{dy}{dt} \\ &= f_{xx}|_{(a,b)}(x_0 - a)^2 + 2\, f_{xy}|_{(a,b)}(x_0 - a)(y_0 - b) + f_{yy}|_{(a,b)}(y_0 - b)^2 \end{aligned}$$

and so on. Substituting into (4.8), we obtain for $t = 1$, and for $n = 1$ and $n = 2$, for some $\zeta \in [a, x_0]$ or $[x_0, a]$ and $\eta \in [b, y_0]$ or $[y_0, b]$,

$$\begin{aligned} f(x_0, y_0) &= f(a,b) + f_x(\zeta,\eta)(x_0 - a) + f_y(\zeta,\eta)(y_0 - b), \ (n = 1) \\ f(x_0, y_0) &= f(a,b) + f_x(a,b)(x_0 - a) + f_y(y_0 - b) \\ &\quad + \frac{1}{2}[f_{xx}(\zeta,\eta)(x_0 - a)^2 + 2f_{xy}(\zeta,\eta)(x_0 - a)(y_0 - b) + f_{yy}(\zeta,\eta)(y_0 - b)^2], \ (n = 2) \end{aligned}$$

and similar expression for $n > 2$. In order to express the n^{th} order formula in a compared way, let us define

$$D := (x_0 - a)\frac{\partial}{\partial x} + (y_0 - b)\frac{\partial}{\partial y}$$

so that in (4.8) we have

$$F^{(k)}(0) = D^k f\Big|_{(a,b)} + \cdots + \frac{1}{(n-1)!}D^{n-1}f\Big|_{(a,b)} + R_n;$$

$R_n = \frac{1}{n!}D^n f\Big|_{(\zeta,\eta)}$ for $\zeta \in [a, x]$ or $[x, a]$ and for $\eta \in [b, y]$ or $[y, b]$. Note that the expression.

$$f(x,y) = f(a,b) + f_x(\zeta,\eta)(x - a) + f_y(\zeta,\eta)(y - b)$$

for ζ between a, x and for η between b, y can be viewed as a 2D Mean Value Theorem.

Example

Find a linear approximation to $f(x,y) = e^{x+y} - 3y$ over $0.8 \le x \le 1.2$ and $-0.1 \le y \le 0.1$ and give bound on the error of approximation, we have

$$f_x = e^{x+y}, \ f_y = e^{x+y} - 3, \ f_{xx} = f_{xy} = f_{yx} = f_{yy} = e^{x+y}$$

Let $a = 1$, $b = 0$ (midpoints of the given intervals). Then

$$\begin{aligned} f(x,y) &\cong f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) \\ &= e + e(x-1) + (e-3)y \\ &= ex + (e-3)y \end{aligned}$$

is a linear approximation with error

$$R_2 = \frac{1}{2}[e^{\zeta+\eta}][(x-1)^2 + 2(x-1) + y^2]$$

for some $\zeta \in [0.8, 1.2]$ and $\eta \in [-0.1, 0.1]$, Then

$$|R_2| \leq e^{1.3}[0.0045]$$

Example
The first two terms (linear and quadratic) of the Taylor Series expansion if e^{xy} about $(a,b) = (1,2)$ is

$$e^{xy} = e^2 + 2e^2(x-1) + e^2(y-2) + 2e^2(x-1)^2 + 3e^2(x-1)(y-2) + \frac{1}{2}e^2(y-2)^2 + \dots$$

as $f_x = ye^{xy}$, $f_y = xe^{xy}$, $f_{xx} = y^2e^{xy}$, $f_{xy} = (1+xy)e^{xy}$, $f_{yy} = x^2e^{xy}$ etc. Note that

$$R_1 = (x-1)\eta e^{\zeta\eta} + (y-2)\zeta e^{\zeta\eta}$$

$$R_2 = \frac{1}{2}[(x-1)\eta^2 + 2(x-1)(y-2)\zeta\eta + (y-2)^2\zeta^2]e^{\zeta\eta}$$

4.6 Implicit Functions and Jacobians

An equation $f(x,y) = 0$ can be viewed as defining y as a function of x implicitly so that it can be written as

$$f(x, y(x)) = 0$$

Example
$x^2 + 4y^2 - 4 = 0 \implies y = \pm\sqrt{1 - \frac{x^2}{4}}$. We have two functions of x defined by $f(x,y) = 0$, which are upper and lower part of ellipse $\frac{x^2}{4} + y^2 = 1$. The domain of definition of both are $-2 \leq x \leq 2$. If we are interested in $3 \leq x \leq 8$, then $f(x,y) = 0$ defines no function.

More often than not, it is not possible to solve for y explicitly, as in $2xy + \sin y = 3$, which is a transcendental equation.

Definition: A function f is of class C^1, denoted as $f \in C^1$, if f_x and f_y exist and we are continuous in an open set, or open domain. It is of class C^2, $f \in C^2$, if f_x, f_y, f_{xx}, f_{xy}, f_{yx}, f_{yy} exist and are continuous on an open set.

4.6.1 Implicit Function Theorem

Let $f(x_0, y_0) = 0$ and suppose $f(x, y)$ is C^1 in a neighbourhood of (x_0, y_0) with

$$f_y(x_0, y_0) \neq 0$$

Then, $f(x, y) = 0$ uniquely defines $y(x)$ in some neighbourhood N of x_0 such that $y(x_0) = y_0$ and $\dot{y}$ exists in N.

Example

Let $(x_0, y_0) = (1, \frac{-\sqrt{3}}{2})$ and consider $x^2 + 4y^2 - 4 = 0$, we have $f_x(x, y) = 2x,\ f_y(x, y) = 8y$, which are continuous everywhere so that $f \in C^1$ everywhere and hence in a neighbourhood of $(1, -\frac{\sqrt{3}}{2})$. Further,

$$f_y\left(1, -\frac{\sqrt{3}}{2}\right) = -4\sqrt{3} \neq 0$$

so that by the theorem, $y(x)$ exists in a neighbourhood of $x_0 = 1$. In fact, $y = -\sqrt{1 - \frac{x^2}{4}}$ is defined everywhere in $(-2, 2) \subseteq \mathbb{R}$.

Example

$f(x, y) = (y - 2x)e^y - x^2 + 1 = 0,\ (x_0, y_0) = (1, 2)$, are such that $f(1, 2) = 0$ and $f_x = -2e^y - 2x,\ f_y = (y - 2x + 1)e^y$. Thus, $f \in C^1$ everywhere. Also, $f_y(1, 2) = e^2 \neq 0$ so that we know that $y(x)$ exists in a neighbourhood of $x_0 = 1$. However, an explicit expression for $y(x)$ is not clear. The theorem only tells the existence

How to compute $y(x)$ when it exists: Suppose $y(x)$ exists about x_0. We can consider its Taylor expansion about x_0:

$$y(x) = y(x_0) + y'(x_0)(x - x_0) + \frac{y''(x_0)}{2}(x - x_0)^2 + \dots \tag{4.9}$$

provided y'' exists about x_0, etc. We also have

$$\frac{d}{dx} f(x, y(x)) = f_x(x, y) + f_y(x, y)y' = 0$$

which gives, provided $f_y(x_0, y_0) \neq 0$, that

$$y'(x_0) = -\frac{f_x(x_0, y_0)}{f_y(x_0, y_0)}$$

Now, if $f_y(x, y) \neq 0$ in a neighbourhood of (x_0, y_0) and if $f \in C^2$, then

$$\begin{aligned} y'' &= -\frac{(f_{xx} + f_{xy}y')f_y - (f_{yy}y' + f_{yx})f_x}{f_y^2} \\ &= \frac{f_x^2 f_{yy} - f_y^2 f_{xx} + 2f_x f_y f_{xy}}{f_y^3} \end{aligned}$$

Continuing this way, the terms of the Taylor series (4.9) can be computed from the partial derivation of f and its evaluations at (x_0, y_0)

Example: $f(x, y) = (y - 2x)e^y - x^2 + 1,\ (x_0, y_0) = (1, 2)$

$$y'(1) = -\frac{f_x(1,2)}{f_y(1,2)} = -\frac{-2e^2 - 2}{e^2} = 2(1 + e^{-2})$$

writing $y' = \dfrac{2 + 2xe^{-y}}{y - 2x + 1}$ and differentiating, we get

$$\begin{aligned} y'' &= \frac{(2e^{-y} - 2xe^{-2}y')(y - 2x + 1) - (y' - 2)(2 + 2xe^{-y})}{(y - 2x + 1)^2} \\ y''(1) &= \frac{2e^{-}2 - 2e^{-2}2(1 + e^{-2}) - 2e^{-2}(2 + 2e^{-2})}{1} = -6e^{-2} - 8e^{-4} \end{aligned}$$

Thus,

$$y(x) = 2 + 2(1 + e^{-2})(x - 1) + (-3e^{-2} - 4e^{-4})(x - 1)^2 + \dots$$

Example: Implicit function theorem can be used to obtain inverse functions, Consider $y = \sin(x)$ and let us write

$$f(\hat{x}, \hat{y}) = \hat{x} - \sin(\hat{y}) = 0$$

and note that $\hat{y}(\hat{x})$ would be the inverse function of $\sin(\hat{x})$. Let us choose $\hat{x}_0$ and consider $(\hat{x_0}, \hat{y_0}) = (0, 0)$ that satisfies $f(\hat{x_0}, \hat{y_0}) = 0$. Also, $f \in C^1$ since $f_{\hat{x}} = 1,\ f_{\hat{y}} = -\cos(\hat{y})$. Further, $f_{\hat{y}}(0, 0) = -\cos(0) = -1 \neq 0$. By the Theorem, $\hat{y}(\hat{x})$ exists for $\hat{x}$ in a neighbourhood of $\hat{x_0} = 0$. We actually know that it exists in $(-1, 1) \subseteq \mathbb{R}$:*insert images here*

4.6.2 Multivariable Case: Jacobian

Consider a system of n-equations

$$\begin{aligned} f_1(x_1, \dots, x_n, u_1, \dots, u_n) &= 0 \\ &\vdots \\ f_n(x_1, \dots, x_n, u_1, \dots, u_n) &= 0 \end{aligned} \tag{4.10}$$

Does there exist $u_1(x_1, \dots, x_n), \dots, u_n(x_1, \dots, x_n)$, functions of n variables, satisfying these equations? Let $P = (x_{10}, \dots, x_{n0}, u_{10}, \dots, u_{n0})$ be a point on $2n$-space such that (4.10) holds at P

Fact: If $f_1, \dots, f_n \in C^1$ in some neighbourhood of P if

$$\det \left.\begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \cdots & \frac{\partial f_1}{\partial u_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial_n f}{\partial u_1} & \cdots & \frac{\partial f_n}{\partial u_n} \end{bmatrix}\right|_P \neq 0$$

then there exists C^1 functions $u_1(x_1,\ldots,x_n),\ldots,u_n(x_1m,\ldots,x_n)$ that satisfy (4.10) in some neighbourhood of $(x_{10},\ldots,x_{n0})$ in n-space.

Example:

$$\left.\begin{array}{r} x = r\cos(\theta) \\ y = r\sin(\theta) \end{array}\right\} \begin{array}{l} f_1(x,y,r,\theta) = x - r\cos(\theta) = 0 \\ f_2(x,y,r,\theta) = y - r\sin(\theta) = 0 \end{array}$$

All particular derivatives of f_1 and f_2 exists and are continuous in $\mathbb{R}^4 = \{(x,y,r,\theta)\}$. Further

$$\det\begin{bmatrix} \dfrac{\partial f_1}{\partial r} & \dfrac{\partial f_1}{\partial \theta} \\ \dfrac{\partial f_2}{\partial r} & \dfrac{\partial f_2}{\partial \theta} \end{bmatrix} = \det\begin{bmatrix} -\cos(\theta) & r\sin(\theta) \\ -\sin(\theta) & -r\cos(\theta) \end{bmatrix} = r$$

Which is non-zero everywhere in $\mathbb{R}^4$ except when $r = 0$. It follows by the Fact that, the functions $r(x,y)$, $\theta(x,y)$ exist in any neighbourhood excluding the origin, i.e., the point $(0,0)$ in xy-plane. We denote

$$J(u_1,\ldots,u_n) = \frac{\partial(f_1,\ldots,f_n)}{\partial(u_1,\ldots,u_n)} = \det\begin{bmatrix} \dfrac{\partial f_1}{\partial u_1} & \cdots & \dfrac{\partial f_1}{\partial u_n} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial f_n}{\partial u_1} & \cdots & \dfrac{\partial f_n}{\partial u_n} \end{bmatrix}$$

and call it the Jacobian if f with respect to u. The Jacobian comes up in the calculating $\dfrac{\partial u_j}{\partial x_i}$ for $i,j = 1,\ldots,n$.

Example:

$$\left.\begin{array}{r} f(x,y,u,v) = 0 \\ g(x,y,u,v) = 0 \end{array}\right\} \xrightarrow[\substack{\text{If } u \text{ and } v \\ \text{exists as functions} \\ \text{of } (x,y)}]{} \begin{array}{l} F(x,y) = f(x,y,u(x,y),v(x,y)) = 0 \\ G(x,y) = g(x,y,u(x,y),v(x,y)) = 0 \end{array}$$

Computing $u(x,y)$, $v(x,y)$ requires terms in the Taylor series, e.g.

$$u(x,y) = u(x_0,y_0) + u_x(x_0,y_0)(x-x_0) + u_y(x_0,y_0,)(y-y_0) + \ldots$$

Now,

$$\begin{aligned} \frac{\partial F}{\partial x} &= f_x + f_u u_x + f_v v_x = 0, \ \frac{\partial F}{\partial y} = f_y + f_u u_y + f_v v_y = 0 \\ \frac{\partial G}{\partial x} &= g_x + g_u u_x + g_v v_x = 0, \ \frac{\partial G}{\partial y} = g_x + g_u u_y + g_v v_y = 0 \end{aligned}$$

so that

$$-\begin{bmatrix} f_x \\ g_x \end{bmatrix} = \begin{bmatrix} f_u & f_v \\ g_u & g_v \end{bmatrix} \begin{bmatrix} u_x \\ v_x \end{bmatrix}, \; -\begin{bmatrix} f_y \\ g_y \end{bmatrix} = \begin{bmatrix} f_u & f_v \\ g_u & g_v \end{bmatrix} \begin{bmatrix} u_y \\ v_y \end{bmatrix}$$

$$u_x = -\frac{\det \begin{bmatrix} f_x & f_v \\ g_x & g_v \end{bmatrix}}{\det \begin{bmatrix} f_u & f_v \\ g_u & g_v \end{bmatrix}}, \; v_x = -\frac{\det \begin{bmatrix} f_u & f_x \\ g_u & g_x \end{bmatrix}}{\det \begin{bmatrix} f_u & f_v \\ g_u & g_v \end{bmatrix}} \text{ or,}$$

$$u_x = -\frac{\frac{\partial(f,g)}{\partial(x,v)}}{\frac{\partial(f,g)}{\partial(u,v)}}, \; v_x = -\frac{\frac{\partial(f,g)}{\partial(u,x)}}{\frac{\partial(f,g)}{\partial(u,v)}}$$

and similarly for u_y and v_y. Since $J(u,v)$ occur in the denominator of every partial derivative of u, v it must be non-zero for the existence of every partial derivative u_x, u_y, v_x, v_y, etc.

4.7 Maxima and Minima of Functions

4.7.1 1-D Case

A function $f(x)$ has an absolute maximum at x_0 if $f(x) \leq f(x_0)$ for all x in the domain of definition of f. It has an absolute minimum at x_0 if $f(x_0) \leq f(x)$ for all x in the domain of definition of f. It has a local maximum at x_0 if $f(x) \leq f(x_0)$ for all x in a neighbourhood of x_0 and a local minimum if $f(x) \geq f(x_0)$ for all x in a neighbourhood. In the figure *insert figure*, A, B, C, D, F are also called extremum points (max or min points) but not E, which is called and inflection point.

Fact: If $f(x)$ has a local extremum at a point x_0, and if $f'(x_0)$ exists, then

$$f'(x_0) = 0 \tag{4.11}$$

Proof: Suppose x_0 is a local minimum point so that $f(x_0) \leq f(x)$ i a neighbourhood $N(x_0)$. Then, as

$$f'(x) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

the left limit is negative and the right limit is positive as $x \to x_0$. It must be that $f'(x_0) = 0$. The same argument applies to a local maximum with small alterations. It follows that (4.11) is a necessary condition. t is not sufficient since the point E in the above example also satisfies $f'(E) = 0$ (zero slope), whereas E is neither a local maximum nor a local minimum. Such a point as E is called horizontal inflection point. A sufficient condition is given next

Theorem: Suppose that for some x_0 in the domain of definition of fm we have that, for $n \geq 2$,

$$f'(x_0) = 0, \; f''(x_0) = 0, \ldots, f^{(n-1)}(x_0) = 0 \text{ but } f^{(n)}(x_0) \neq 0 \tag{4.12}$$

Then if $f^{(n)}(x)$ is continuous in a neighbourhood of x_0,

(i) n even and $f^{(n)}(x_0) < 0 \implies x_0$ is local maximum point.

(ii) n even and $f^{(n)}(x_0) > 0 \implies x_0$ is local minimum point.

(iii) n odd $\implies x_0$ is horizontal inflection point

Proof: As $f^{(n)}(x)$ is continuous inn some neighbourhood $N(x_0)$, we have

$$\text{sign } f^{(n)}(x) = \text{sign } f^{(n)}(x_0), \ \forall x \in N(x_0)$$

By Taylor's formula

$$f(x) = f(x_0) + \frac{f^{(n)}(\zeta)}{n!}(x - x_0)^n, \ x \in N(x_0) \tag{4.13}$$

for some $\zeta \in \tilde{N}(x_0) \subseteq N(x_0)$, where $\tilde{N}(x_0)$ is a smaller neighbourhood. Thus, sign $f^{(n)}(\zeta)$ =sign $f^{(n)}(x_0)$. It follows by (4.13) that, when n is even

$$\text{sign}[f(x) - f(x_0)] = \text{sign } f^{(n)}(x_0), \ \forall x \in \tilde{N}(x_0)$$

which gives the first two claims. If, on the other hand, n is odd, then depending on the whether $x < x_0$ and $x > x_0$, we will have $f(x) - f(x_0)$ have opposite signs, which gives that x_0 is an inflection point.

Example: $f(x) = (2\sqrt{x} - x - 1)^2 = (\sqrt{x} - 1)^4; \ 0 < x < \infty$. Note that $f'(1) = 0, \ f''(1) = 0, \ f'''(1) = 0, f^{(4)}(1) = \frac{3}{2} \neq 0$. Since n is even and $f^{(4)}(1) > 0$, we have a local minimum at $x_0 = 1$.

Example: $f(x) = (x - 1)^2 \ln(x); \ 0 < x < \infty$. Now,

$$\begin{aligned}
f'(x) &= (x - 1)[2\ln(x) + \frac{x - 1}{x}] \\
f''(x) &= 2\ln(x) + 2\frac{x - 1}{x} - \frac{(x - 1)^2}{x^2} + \frac{2}{x}(x - 1) \\
f'''(x) &= \frac{2}{x} + \frac{2}{x^2} + \frac{2}{x^3}(x - 1)^3 - \frac{2}{x^2}(x - 1) - \frac{2}{x^2}(x - 1) + \frac{2}{x}
\end{aligned}$$

so that, $f'(1) = 0, \ f''(1) = 0, \ f'''(1) = 6 \neq 0$. Hence, n is odd so that there is a horizontal inflection point at $x_0 = 1$. Note that $f'(x) = 0$ has only one solution $x_0 = 1$ since the curves $\ln(x)$ and $\frac{1 - x}{2x}$ intersects only at $x = x_0 = 1$

4.7.2 Multivariable Case

Let us consider $f(\underline{x}) = f(x_1, \ldots, x_2)$ defined for $\underline{x} \in D \subseteq \mathbb{R}^n$ and a point $\hat{\underline{x}} = (\hat{x_1}, \ldots, \hat{x_n}) \in D$. If there is a neighbourhood $N(\hat{\underline{x}})$ such that

$$\begin{aligned}
f(\underline{x}) \leq f(\hat{\underline{x}}) \ \forall \underline{x} \in N(\hat{\underline{x}}) &\implies \hat{\underline{x}} \text{ is a local maximum point} \\
f(\underline{x}) \geq f(\hat{\underline{x}}) \ \forall \underline{x} \in N(\hat{\underline{x}}) &\implies \hat{\underline{x}} \text{ is a local minimum point}
\end{aligned}$$

of f.

Theorem: If f has a local extremum at $\underline{\hat{x}} \in D$ and if $f \in C^1$ (continuously differentiable) in a neighbourhood of $\underline{\hat{x}}$, then

$$\frac{\partial f}{\partial x_1} = 0, \ldots, \frac{\partial f}{\partial x_n} = 0 \tag{4.14}$$

Proof: Consider any curve passing through $\underline{\hat{x}}$ in n-space and let $x_1 = x_1(\tau), \ldots, x_n = x_n(\tau)$ be a parametric equation of this curve such that $\underline{x}(0) = \underline{\hat{x}}$, i.e.

$$x_1(0) = \hat{x_1}, \ldots, x_n(0) = \hat{x_n}$$

By $f \in C^1$, whenever the chosen curve is smooth, i.e., $x_j'(\tau)$ exists in an interval about $\tau = 0$, we have that the composition function

$$F(\tau) = f(\underline{x}(\tau)) = f(x_1(\tau), \ldots, x_n(\tau))$$

is differentiable in this interval of $\tau = 0$ with

$$\frac{dF}{dz}(0) = \frac{\partial f}{\partial x_1}(\underline{\hat{x}})\frac{dx_1}{d\tau}(0) + \cdots + \frac{\partial f}{\partial x_n}(\underline{\hat{x}})\frac{dx_n}{d\tau}(0) = 0 \tag{4.15}$$

for every choice of such smooth curves. As the chosen curves through $\underline{\hat{x}}$ changes, the values $\frac{dx_j}{d\tau}(0)$ assume infinitely many different sets. Hence, the only way (4.15) holds for all these curves is that (4.14) is satisfied. A point $\underline{\hat{x}}$ satisfying (4.14) is called a critical point of f_0. Not all critical points are local extrema, just like in 1-D case.

Example: $f(x_1, x_2) = x_1^2 - x_2^2$

$$\frac{\partial f}{\partial x_1} = 2x_1, \ \frac{\partial f}{\partial x_2} = -2x_2$$

which are both zero at $(\underline{\hat{x_1}}, \underline{\hat{x_2}}) = (0, 0)$. However, $f > 0$ for $x_1 > x_2$ and $f < 0$ for $x_1 < x_2$ and in any neighbourhood of $(0, 0)$ there are infinitely many such points. Hence $\underline{\hat{x}} = \underline{0}$ is neither a local maximum nor a local minimum points. It is called a saddle point.

Theorem: (Classification of Critical Points) Consider that

$$f_{x_1}(\underline{\hat{x}}) = 0, \ldots, f_{x_n}(\underline{\hat{x}}) = 0$$

for some $\underline{\hat{x}}$ in the domain of f. If $f \in C^2$ in some $N(\underline{\hat{x}})$ and if $\det \hat{H}(\underline{\hat{x}}) \neq 0$, where

$$H = \begin{bmatrix} f_{x_1x_1}(\underline{\hat{x}}) & \cdots & f_{x_1x_n}(\underline{\hat{x}}) \\ \vdots & \ddots & \vdots \\ f_{x_nx_1}(\underline{\hat{x}}) & \cdots & f_{x_nx_n}(\underline{\hat{x}}) \end{bmatrix} \text{ (Hessian)}$$

(i) H is positive definite $\implies$ $\underline{\hat{x}}$ is a local minimum point.

(ii) H is negative definite $\implies$ $\underline{\hat{x}}$ is a local maximum point.

(iii) H has at least one positive and one negative eigenvalue (Indefinite) $\implies$ $\underline{\hat{x}}$ is a saddle point.

(iv) Otherwise, the Hessian does not give sufficient information to classify $\underline{\hat{x}}$.

Remark: Multivariable Taylor's formula can be written as

$$f_{\underline{x}} = f(\underline{\hat{x}}) + J(\underline{\hat{x}})(\underline{x} - \underline{\hat{x}}) + \frac{1}{2}(\underline{x} - \underline{\hat{x}})^T H(\underline{\hat{x}})(\underline{x} - \underline{\hat{x}}) + \dots$$

where J is the Jacobian (row-vector $[f_{x_1}, \dots, f_{x_n}]$) and H is the Hessian.

Recall: A complex $\lambda \in \mathbb{C}$ is an eigenvalue of a matrix M if it a root of the characteristic polynomial

$$\Delta(s) = \det(sI - M)$$

If M is symmetric, then all its eigenvalues are real numbers. A symmetric M is positive definite if all its eigenvalues are positive. It is positive semi-definite if all its eigenvalues are non-negative. A matrix M is negative definite if it is symmetric and if $-M$ is positive definite. Similarly for negative semi-definite.

Example

(1)

$$M = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \; \lambda I - M = \begin{bmatrix} \lambda - 1 & 1 \\ 1 & \lambda - 1 \end{bmatrix}$$

$\det(\lambda I - M) = (\lambda - 1)^2 - 1 = 0 \implies \lambda_1 = 0, \; \lambda_2 = 2.$ so that M is positive semi-definite.

(2)

$$M = \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}$$

$\det(\lambda I - M) = (\lambda - 1)^2 - 4 = 0 \implies \lambda_1 = 3, \; \lambda_2 = -1$, so that M is neither positive nor negative (semi)definite, thus indefinite.

(3)

$$M = \begin{bmatrix} 1 & \frac{-1}{2} \\ \frac{-1}{2} & 1 \end{bmatrix}$$

$\det(\lambda I - M) = (\lambda - 1)^2 - \dfrac{1}{4} = 0 \implies \lambda_1 = \dfrac{1}{2}, \; \lambda_2 = \dfrac{3}{2}$ so that M is positive definite.

Example: $f(x, y) = \ln 2x(y-1) + 1 \implies f_x = \dfrac{2(y-1)}{2x(y-1)+1}, \; f_y = \dfrac{2x}{2x(y-1)+1}$. Setting $f_x = 0, \; f_y = 0$, we have $\underline{\hat{x}} = (0, 1)$ is a critical point.

$$H(\underline{\hat{x}}) = \left. \begin{bmatrix} \dfrac{-4(y-1)}{[2x(y-1)+1]^2} & \dfrac{2}{[2x(y-1)+1]^2} \\ \dfrac{2}{[2x(y-1)+1]^2} & \dfrac{-4x^2}{[2x(y-1)+1]^2} \end{bmatrix} \right|_{\underline{x} = \underline{\hat{x}}} = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$$

Eigenvalues of $H(\underline{\hat{x}})$ are found from $\lambda^2 - 4 = 0$ as $\{2, -2\}$. The point $\underline{\hat{x}} = (0, 1)$ is a saddle point of $f(x, y)$.

4.7.3 Constrained Extrema

Consider the problem of finding the extrema for $f(x_1, \dots, x_n)$ subject to the constraints $g_j(x_1, \dots, x_n) = 0;\ j = 1, \dots, k$

Example: Determine the closest point on the plane $2x - y + z = 3$ to the origin. Here $f(x, y, z) = x^2 + y^2 + z^2$, distance to the origin of a point (x, y, z) ad the only constraint is $g(x, y, z) = 2x - y + z - 3 = 0$ so that $n = 3$, $k = 1$. Such a problem can be solved by converting it ti and unconstrained problem of finding the extrema for

$$f^*(\underline{x}, \underline{\lambda}) = f(\underline{x}) - \lambda_1 g_1(\underline{x}) - \cdots - \lambda_k g_k(\underline{x})$$

where $\underline{\lambda} = (\lambda_1, \dots, \lambda_k)$

Example: In the example above

$$f^*(x, y, z, \lambda) = x^2 + y^2 + z^2 - \lambda(2x - y + z - 3)$$

so that

$$\begin{aligned} \frac{\partial f^*}{\partial x} &= 2x - 2\lambda, \ \frac{\partial f^*}{\partial y} = 2y + \lambda \\ \frac{\partial f^*}{\partial z} &= 2z - \lambda, \ \frac{\partial f^*}{\partial \lambda} = 2x - y + z - 3 \end{aligned}$$

Setting all these to zero, we get

$$\begin{bmatrix} 2 & 0 & 0 & -2 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 2 & -1 \\ 2 & -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 3 \end{bmatrix} \implies \begin{bmatrix} x \\ y \\ z \\ \lambda \end{bmatrix} = \begin{bmatrix} 1 \\ -\frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

Hence, $\hat{\underline{x}} = (1, -1/2, 1/2)$ is a critical point and form the geometry of the problem, must be the minimizing point on the plane.

Remark: The Hessian matrix sufficient condition does not apply to constrained problems via Lagrange multiplier solution. A second and more obvious method is the method of elimination, where constraints used to reduce the dimension of the problem to functions of less number of variables.

Example: Using the constraint $2x - y + z = 3$ we have that

$$y = -3 + 2x + z$$

so that $F(x, z) = f(x, y, z) = f(x, -3 + z + 2x, z) = x^2 + (3 - z - 2x)^2 + z^2$. Taking derivatives of F, we have

$$\left.\begin{aligned} \frac{\partial F}{\partial x} &= 2x - 4(3 - 2x - z) = 0 \\ \frac{\partial F}{\partial z} &= -2(3 - 2x - z) + 2z = 0 \end{aligned}\right\} \begin{bmatrix} 10 & 4 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ 6 \end{bmatrix}$$

which gives $(\hat{x}, \hat{z}) = (1, 1/2)$ and $\hat{y} = -3 + 2\hat{x} + \hat{z} = -1/2$. Hence, $\underline{\hat{x}} = (1, -1/2, 1/2)$ is a critical point, which by

$$\begin{bmatrix} 10 & 4 \\ 4 & 4 \end{bmatrix}, \ \det(H) = 24 > 0$$

is a minimum point.

<u>Remark:</u> Given $g_j(x_1, \ldots, x_n) = 0$, we can solve for the implicitly defined $x_n = G(x_1, \ldots, x_{n-1})$ if $\dfrac{\partial g_j}{\partial x_n}$ is non-zero at a point $(x_{10}, \ldots, x_{n-1,0})$. The function G is defined in a neighbourhood of $(x_{10}, \ldots, x_{n-1,0})$.

<u>Remark:</u> In $2D$, we can visualize minimize $f(x, y)$ subject to $g(x, y) = c$ as follows: If $(\hat{x}, \hat{y})$ is a solution, then minimum is d_2 in the figure, where the level curves and the curve $g(x, y) = c$ are shown *insert image here*. Thus, at $(\hat{x}, \hat{y})$, gradients of f and g are parallel, i.e., there is λ such that

$$\nabla f = \lambda \nabla g \text{ or } f_x - \lambda g_x = 0, \ f_y - \lambda g_y = 0$$

Method of Lagrange

Minimize $f(x, y, z)$ subject to $g(x, y, z) = 0$. We have

$$df = f_x dx + f_y dy + f_z dz = 0$$
$$dg = g_x dx + g_y dy + g_z dz = 0$$

which give

$$df - \lambda dg = (f_x - \lambda g_x)dx + (f_y - \lambda g_y)dy + (f_z - \lambda g_z)dz = 0$$

for all λ. Since dx, dy, dz are not independent increments, we cannot conclude that $f_x - \lambda g_x = 0$, etc. However, note that if, e.g., $g_z \neq 0$ at the critical point $(\hat{x}, \hat{y}, \hat{z})$, then leaving

$$\lambda = \frac{f_z(\hat{x}, \hat{y}, \hat{z})}{g_z(\hat{x}, \hat{y}, \hat{z})}$$

we obtain $(f_x - \lambda g_x)dx + (f_y - \lambda g_y)dy = 0$ at $(\hat{x}, \hat{y}, \hat{z})$, where dx and dy are now independent increments. This gives

$$f_x(\hat{x}, \hat{y}, \hat{z}) = \lambda g_x(\hat{x}, \hat{y}, \hat{z}) \text{ and } f_y(\hat{x}, \hat{y}, \hat{z}) = \lambda g_y(\hat{x}, \hat{y}, \hat{z})$$

as well.

Method of Elimination

Suppose $g_z(\hat{x}, \hat{y}, \hat{z}) \neq 0$ so that in a neighbourhood of $(\hat{x}, \hat{y}, \hat{z})$, we have $z = G(x, y)$ for a function G given by the implicit function theorem. Then,

$$f(x, y, G(x, y)) = F(x, y)$$

is a well defined composite function in a neighbourhood of $(\hat{x}, \hat{y})$. Since

$$df = \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy = 0 \text{ } (df = 0 \text{ at an extremum})$$

for independent dx and dy, we can conclude that

$$F_x(\hat{x}, \hat{y}) = 0, \ F_y(\hat{x}, \hat{y}) = 0$$

Chapter 5

Functions of a Complex Variable

A function $w = w(z)$ of a complex variable z is a rule that assigns a unique value $w(z) \in \mathbb{C}$. Writing $w(z) = u + iv$ for $u, v \in \mathbb{R}$, and $z = x + iy$ for $x, y \in \mathbb{R}$ we have

$$w(z) = u(x, y) + iv(x, y)$$

the real and imaginary parts of w are functions of two variables x and y.

Example
$w = z^2$, $\mathbb{D}$ =First Quadrant= $z = x + iy$; $k \geq 0$, $y \geq 0$. We can not plot w for all $z \in \mathbb{D}$ but can have an idea by plotting the image of certain geometric objects under the mapping $w = z^2$

$$L_1 : y = y_0,\ 0 \leq x \leq \infty \implies u = x^2 - y_0^2,\ v = 2xy_0 \implies u = \left(\frac{v}{2y_0}\right)^2 - y_0^2$$

$$L_2 : x = x_0,\ 0 \leq y \leq \infty \implies u = x_0^2 - y^2,\ v = 2x_0 y \implies u = x_0^2 - \left(\frac{v}{2x_0}\right)^2$$

5.1 Elementary Functions

5.1.1 Exponential Function

Recall that $e^{iy} = \cos y + i \sin y$ was the Euler's formula. Exponential function is defined as

$$e^z = e^x(\cos y + i \sin y) = e^x e^{iy} = e^{x+iy}$$

Note that its modulus is given by

$$|e^z| = e^x \sqrt{\cos^2 y + \sin^2 y} = e^x$$

and that

$$e^z = e^{z+i2\pi k},\ \forall k \in \mathbb{Z}$$

The exponential function of a complex variable is hence a bit different from that of a real variable, although similar. For example $\frac{d}{dz} e^z = e^z$, as we will later show.

5.1.2 Trigonometric and Hyperbolic Functions

We can write, by Euler's formula, $\cos y = \frac{1}{2}[e^{iy} + e^{-iy}]$ and $\sin y = \frac{1}{2i}[e^{iy} - e^{-iy}]$. This motivates the definitions

$$\cos z = \frac{e^{iz} + e^{-iz}}{2},\ \sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

Similarly

$$\cosh z = \frac{e^{z} + e^{z}}{2},\ \sinh z = \frac{e^{z} - e^{z}}{2}$$

Observe that

$$\cos(iz) = \cosh z,\ \sin(iz) = i\sinh z$$

$$\cosh(iz) = \cos z,\ \sinh(iz) = ii\sin z$$

The usual identities for trigonometric and hyperbolic functions also hold:

$$\sin^2 z + \cos^2 z = 1,\ \sin(-z) = -\sin z,\ \cos(-z) = \cos z$$

$$\cosh^2 z - \sinh^2 z = 1,\ \cosh(-z) = \cosh z,\ \sinh(-z) = -\sinh z$$

Also,

$$\frac{d}{dz}\sin z = \cos z,\ \frac{d}{dz}\cos z = -\sin z,\ \frac{d}{dz}\cosh z = \sinh z,\ \text{etc.}$$

5.1.3 Integrals of Complex Valued Functions

Suppose $f(x) \in \mathbb{C}$ for $x \in \mathbb{R}$. Then we can define

$$\int f(x)dx = \int \Re\{f(x)\}dx + i\int \Im\{f(x)\}dx$$

which means

$$\Re\left\{\int f(x)dx\right\} = \int \Re\{f(x)\}dx,\ \Im\left\{\int f(x)dx\right\} = \int \Im\{f(x)\}dx$$

<u>**Example**</u>

$$\begin{aligned}
\int_0^\infty e^{-x}\cos(ax)dx &= \int_0^\infty \Re\{e^{-x}e^{iax}\}dx \\
&= \Re\left\{\int_0^\infty e^{-(1-ia)x}dx\right\} \\
&= \Re\left\{\frac{e^{-(1-ia)x}}{-(1-ia)}\right\}\Bigg|_0^\infty \\
&= \Re\left\{\frac{1}{1-ia}\right\} - \frac{1}{1-ia}\lim_{x\to\infty}[e^{-x}e^{iax}]
\end{aligned}$$

But, $\lim_{x\to\infty}|e^{-x}e^{iax}| = \lim\limits_{x\to\infty} e^{-x} = 0$, provided $a \in \mathbb{R}$, so that $\lim\limits_{x\to\infty}[e^{-x}e^{iax}] = 0$ as well. Therefore,

$$\int_0^\infty e^{-x}\cos(ax)dx = \Re\left\{\frac{1}{1-ia}\right\} = \frac{1}{1+a^2},\ \forall a \in \mathbb{R}$$

Example

Certain differential equations with sinusoidal forcing functions, like $F\cos(\omega t)$ or $F\sin(\omega t)$, can be solved by finding a particular solution to the forcing function $Fe^{i\omega t}$

$$\begin{aligned} L\ddot{i} + R\dot{i} + \frac{1}{C}i &= \frac{dE(t)}{dt},\ E(t) = E_0\sin(\omega t) \\ \implies L\ddot{i} + R\dot{i} + \frac{1}{C}i &= E_0\omega\cos(\omega t) \end{aligned} \tag{5.1}$$

Let us consider $L\ddot{x} + R\dot{x} + \frac{1}{C}x = E_0\omega e^{i\omega t}$ instead and note that a (possibly complex valued solution) $x(t)$ is such that $i(t) = \Re x(t)$ is a solution to (5.1). Thus a particular solution $x_p(t) = Ae^{i\omega t}$ can be sought according to the method of undetermined coefficients. Then

$$\left[L(i\omega)^2 + R(i\omega) + \frac{1}{C}\right]Ae^{i\omega t} = \omega E_0 e^{i\omega t}$$

is obtained by substitution into $L\ddot{x}_p + R\dot{x}_p + \frac{1}{C}x_p = E_0\omega e^{i\omega t}$ so that

$$x_p(t) = \frac{\omega E_0}{\frac{1}{C} - L\omega^2 + iR\omega}e^{i\omega t};\ i_p(t) = \Re\left\{\frac{\omega E_0 e^{i\omega t}}{\frac{1}{C} - L\omega^2 + iR\omega}\right\}$$

Now,

$$i_p(t) = \Re\left\{\frac{wE_0C[(1 - LC\omega^2)\cos(\omega t) + RC\omega\sin(\omega t)]}{(1 - LC\omega^2)^2 + (RC\omega)^2}\right\}$$

Another expression for this particular solution for (5.1) is

$$i_p(t) = M\sin(\omega t + \phi);\ M := \frac{\omega E_0 c}{\sqrt{(1 - LC\omega^2)^2 + (RC\omega)^2}},\ \phi := \tan^{-1}\left(\frac{1 - LC\omega^2}{RC\omega}\right)$$

5.2 Polar Representation

The polar representation of a complex number $z = x + iy$ is given by $z = re^{i\theta}$, where $r = |z| = \sqrt{x^2 + y^2}$ and $\theta = \arg z = \tan^{-1}\frac{y}{x}$. Note that the and $\theta = \arg z$ is only determined upto a multiple of 2π since $e^{i(\theta + 2\pi k)} = e^{i\theta}$ for any integer k. That value of $\arg z = \theta$, which satisfies $-\pi < \theta \leq \pi$ is called the principal argument of z and denoted by $\operatorname{Arg} z$. Thus, $\arg z$ and $\operatorname{Arg} z$ are related by

$$\arg z = \operatorname{Arg} z + 2\pi k,\ k \in \mathbb{Z}$$

Example

$$z = 1 + i = \sqrt{2}e^{\pi/4} \text{ and } \operatorname{Arg} z = \frac{\pi}{4}$$

All of $\frac{\pi}{4} + 2\pi k$ are valid values for $\arg z$.

Remark: In computing $\theta = \tan^{-1}(\frac{y}{x})$ we also need information about the quadrant in which (x, y) lies. For instance with $z = 1 + i$, $\tan^{-1} 1 = \frac{\pi}{4} + 2\pi k$, not $\tan^{-1} = \frac{5\pi}{4}$ or $\frac{5\pi}{4} + 2\pi k$, because $(x, y) = (1, 1)$ is in the first quadrant.

Convention: The principal argument of a negative real number is taken as π (not $-\pi$)! Also observe that the polar form of $z \in \mathbb{C}$ exists only if $z \neq 0$ since of $z = 0$, the angle is not defined.

The main advantage of polar form is in multiplication and division of complex number. Let $z_1 = r_1 e^{i\theta_1}$, $z_2 = r_2 e^{i\theta_2}$. Then, with $z = re^{i\theta}$,

$$z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)},\ \frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}\ (z_2 \neq 0),$$

$$z^n = (re^{i\theta})^n = r^n e^{in\theta},\ \forall n \in \mathbb{Z} \tag{5.2}$$

By (5.2), $\cos(n\theta) + i\sin(n\theta) = [\cos\theta + i\sin\theta]^n$, called deMoivre's Formula.

5.3 Additional Elementary Functions

5.3.1 Fractional Powers

Given $z \in \mathbb{C}$, let us write the polar form in terms of a fixed $\arg z = \theta_0$ as $z = e^{i\theta_0}$. Then,

$$z^{1/n} = [re^{i(\theta_0+2\pi k)}]^{1/n} = \sqrt[n]{r} e^{i\frac{\theta_0+2\pi k}{n}},$$

where $k = 0, 1, \ldots, n-1$ are values of the integer k that yield distinct points in the plane. These n numbers are the n^{th} roots of z. Here, the positive n^{th} root of the real number r, $\sqrt[n]{r}$ is the modulus of $z^{1/n}$ and the n values $\dfrac{\theta_0 + 2\pi k}{n}$, $k = 0, 1, \ldots, n-1$ are equally spaced angles of all n^{th} roots of z.

Example

$(-8)^{1/3} =$? Let $z = -8 = 8e^{i\pi}$ with $\theta_0 = \pi = \text{Arg}\,{-8}$. Then

$$(-8)^{1/3} = \sqrt[3]{-8} e^{i\frac{(2k+1)\pi}{3}};\ k = 0, 1, 2,$$

so that the three 3^{rd} roots of -8 are

$$\{2e^{i\pi/3}, -2, 2e^{i5\pi/3}\} = \{1 + i\sqrt{3}, -2, 1 - i\sqrt{3}\}$$

5.3.2 Logarithmic Function

We will denote the real logarithmic function by "ln" and the complex logarithmic function by "log". The latter is defined

$$\log z := \log(re^{i\theta}) = \ln(\theta) + i\theta \tag{5.3}$$

which will be as if "blindly" applied the rules for the real logarithmic function. Here, $\theta = \arg z$ so that

$$\log z = \ln r + i(\theta_0 + 2\pi k);\ k \in \mathbb{Z},$$

where θ_0 is any fixed value for $\arg z$, e.g., $\operatorname{Arg} z = \theta_0$

Example

$$\log(1+i) = \log(\sqrt{2}e^{i(\frac{\pi}{4}+2\pi k)}) = \ln\sqrt{2} + i(\frac{\pi}{4} + 2\pi k)$$

Using the definition (5.3), we can also define, with $z = re^{i\theta}$,

$$z^c = e^{c\log z} = e^{c[\ln r + i\theta]}, \ \forall c \in \mathbb{C}$$

Example

$$1^i = e^{i\log 1} = e^{i(\ln 1 + i2\pi k)} = e^{-2\pi k} \text{ for } k \in \mathbb{Z}$$

If we let $k = -20$, then $1^i = e^{40\pi}$, a very large number!

Remark: $e^{\log z} = e^{\ln r + i\theta} = e^{\ln r}e^{i\theta} = re^{i\theta} = z$ whereas, $\log e^z = \ln|e^z| + i\arg e^z$ $= \ln e^x + i(y + 2\pi k) = x + iy + i2\pi k = z + 2\pi k$ for $k \in \mathbb{Z}$

5.4 Branch Cut

Functions $z^{1/n}$ $(n > 1)$, z^c $(c \in \mathbb{C})$, $\log z$ are all multiple valued, not functions in the usual sense! The reason for multiple values is the non-uniqueness of "$\arg z$". For instance different logarithmic functions are obtained by

$$\log^{(a)}(z) := \ln|z| + i\theta, \ -\pi < \theta \le \pi$$
$$\log^{(b)}(z) := \ln|z| + i\theta, \ \pi < \theta \le 3\pi$$

In obtaining these two functions, we specified "branch-cuts" at $\alpha = -\pi$ and $\alpha = \pi$, respectively. If we consider n point like "-2" and $\log(-2)$, then

$$\log^{(a)}(-2) := \ln(2) + i\pi, \ \log^{(b)}(-2) := \ln(2) + i3\pi$$

Moreover, the point A and B in the above figure will have values

$$\log^{(a)}(A) := \ln(2) + i(\pi - \epsilon), \ \log^{(b)}(B) := \ln(2) + i(3\pi - \epsilon)$$

$$\log^{(a)}(A) := \ln(2) + i(-\pi + \epsilon), \ \log^{(b)}(B) := \ln(2) + i(\pi + \epsilon)$$

for a small positive angle ϵ. Similarly, a logarithmic function of branch cut of α would be

$$\log z = \ln|z| + i\arg z, \ \alpha < \arg z \le \alpha + 2\pi$$

so that $\arg C \cong \alpha + 2\pi$ and $\arg D \cong \alpha$ for points C and D in the figure. Going from C to D and passing the branch cut, the argument jumps by an angle of 2π. Thus, the $\log z$, thus defined as a single valued function, is continuous everywhere except along the branch cut.

A suitable branch cut for the function $\log(z - a)$, on the other hand, will be as shown in the figure. We have with $\zeta = z - a$

$$\log(z-a) = \log\zeta = \ln\rho + i\phi; \ \zeta = z - a = \rho e^{i\phi}$$

5.4.1 Inverse Functions

A function $w(z) = \sin^{-1} z$ must satisfy $\sin w = z$ so that

$$z = \frac{e^{i\omega} - e^{-i\omega}}{2i} \implies e^{i\omega} - e^{-i\omega} - 2iz = 0 \implies (e^{i\omega})^2 - 2ize^{i\omega} - 1 = 0$$

which solving for $e^{i\omega}$, gives $e^{i\omega} = iz + \sqrt{1-z^2}$, where $\sqrt{\cdot}$ is the "complex square-root" and has two values. Now applying "log" to both sides, we get

$$i\omega = \log[iz + \sqrt{1-z^2}] \implies \omega = -i\log[iz + \sqrt{1-z^2}]$$

which can be considered to be a "multiple valued inverse function of $\sin z$".

<u>**Example**</u>
$\sin^{-1}(2i) =?$

$$\begin{aligned}
\sin^{-1}(2i) &= -i\log[i(2i) + \sqrt{1-(2i)^2}] \\
&= -i\log(-2+\sqrt{5}) \\
&= -i\log(-2\pm\sqrt{5}) \\
&= \begin{cases} -i[\ln(\sqrt{5}-2) + i2\pi k],\ k \in \mathbb{Z} \\ -i[\ln(\sqrt{5}+2) + i\pi(2l+1)],\ l \in \mathbb{Z} \end{cases} \\
&= \begin{cases} i2\pi k - i\ln(\sqrt{5}-2),\ k \in \mathbb{Z} \\ i\pi(2l+1) - i\ln(\sqrt{5}+2),\ l \in \mathbb{Z} \end{cases}
\end{aligned}$$

Noting that $\sqrt{5}-2 = (\sqrt{5}+2)^{-1}$, we can combine to get

$$\sin^{-1} = n\pi + i(-1)^n \ln(\sqrt{5}+2),\ n \in \mathbb{Z}$$

Other inverse functions, $\cos^{-1} z$, $\sinh^{-1} z$, $\tan^{-1} z$, etc. can be obtained similarly.

5.5 Limits, Continuity, Derivatives

Let $f : D \to \mathbb{C}$ with $D \subseteq \mathbb{C}$ and let $z_0 \in D$. We write $\lim\limits_{z\to z_0} f(z) = L$ if for every $\epsilon > 0$, there is $f > 0$ such that

$$0 < |z - z_0| < \delta \implies |f(z) - L| < \epsilon$$

i.e., if z is δ-close to z_0, then $f(z)$ is ϵ-close to L. This definition is equivalent to that we used for functions of two variables since $|z - z_0| < f$ defines a $N(z_0, \delta)$ (neighbourhood of radius δ about z_0 in xy-plane with $z = x + iy$). If we write $f(z) = u(x, y) + iv(x, y)$, then $\lim\limits_{z\to z_0} f(z) = L$ if and only if

$$\lim_{(x,y)\to(x_0,y_0)} u(x, y) = \Re L \text{ and } \lim_{(x,y)\to(x_0,y_0)} v(x, y) = \Im L$$

where $z_0 = x_0 + iy_0$. This last fact allows us to derive that usual properties of limit holds for the limit of functions of a complex variable. Continuity is obtained if, in addition, $L = f(z_0)$, i.e., if $\lim_{z \to z_0} f(z) = f(z_0)$, in which case we say f is continuous at z_0. The derivative of f at z_0 is defined by

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

whenever the limit exists. Equivalently, if we replace $z - z_0$ by Δz

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z)}{\Delta z}$$

for every Δz small enough so that $f(\Delta z)$ is defined, i.e., $\Delta z \in D$

<u>**Example:**</u> $f(z) = \bar{z}$ is such that

$$\frac{f(z + \Delta z) - f(z)}{\Delta z} = \frac{\overline{z + \Delta z} - \bar{z}}{\Delta z} = \frac{\overline{\Delta z}}{\Delta z} = \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}$$

Now, if $\Delta y = 0$, then the limit is clearly 1 and if $\Delta x = 0$, then, -1, as $\Delta x \to 0$ and $\Delta y \to 0$. The value of the limit depends on the "direction of approach" to z. The derivative, hence, does not exist and we say $f(z) = \bar{z}$ is nowhere differentiable.

<u>**Example:**</u> $f(z) = z^n$, $n \geq 1$ is and integer

$$\begin{aligned}
f'(z) &= \lim_{\Delta z \to 0} \frac{(z + \Delta z)^n - z^n}{\Delta z} \\
&= \lim_{\Delta z \to 0} \frac{\sum_{k=0}^{n} \binom{n}{k} z^k (\Delta z)^{n-k} - z^n}{\Delta z} \\
&= \lim_{\Delta z \to 0} \sum_{k=0}^{n} \binom{n}{k} z^k (\Delta z)^{n-k-1} \\
&= \binom{n}{n-1} z^{n-1} \\
&= n z^{n-1}
\end{aligned}$$

Therefore, z^n is everywhere differentiable with $\frac{d}{dz} z^n = n z^{n-1}$.

5.5.1 Cauchy Riemann Equations

Letting Δz approach 0 first from a horizontal direction and second, from vertical, in xy-plane, we obtain a necessary condition for the existence of $f'(z)$ for any $f(z)$: Let $\Delta z = \Delta x + i\Delta y$, $z = x + iy$, $z_0 = x_0 + iy_0$, and $f(z) = u(x, y) + iv(x, y)$. Then

$$\begin{aligned}
f'(z_0) = \lim_{\Delta z \to 0} &= \frac{f(\Delta z + z_0) - f(z_0)}{\Delta z} \\
= \lim_{\Delta x \to 0, \Delta y \to 0} &\frac{u(x_0 + \Delta x, y + \Delta y) - u(x_0, y_0) + i[v(x_0 + \Delta x, y + \Delta y) - v(x_0, y_0)]}{\Delta x + i\Delta y}
\end{aligned}$$

Letting first $\Delta y = 0$, we have

$$\begin{aligned} f'(z_0) &= \lim_{\Delta x \to 0} \frac{u(x_0 + \Delta x, y_0) - u(x_0, y_0) + i[v(x_0 + \Delta x, y_0) - v(x_0, y_0)]}{\Delta x} \\ &= \left.\frac{\partial u}{\partial x}\right|_{(x_0,y_0)} + i \left.\frac{\partial v}{\partial x}\right|_{(x_0,y_0)} \\ &= u_x(x_0, y_0) + i v_x(x_0, y_0) \end{aligned} \tag{5.4}$$

Letting $\Delta x = 0$, we have

$$\begin{aligned} f'(z_0) &= \left.\frac{\partial v}{\partial y}\right|_{(x_0,y_0)} - i \left.\frac{\partial u}{\partial y}\right|_{(x_0,y_0)} \\ &= v_y(x_0, y_0) - i u_y(x_0, y_0) \end{aligned} \tag{5.5}$$

By equating the real and imaginary parts in (5.4) and (5.5) we get the necessary conditions for $f'(z_0)$ to exist:

$$u_x(x_0, y_0) = v_y(x_0, y_0),\ u_y(x_0, y_0) = -v_x(x_0, y_0) \tag{5.6}$$

These are the Cauchy-Riemann equations.

<u>Theorem:</u>, Let $f(z) = u(x, y) + iv(x, y)$ be defined in a neighbourhood of $z_0 = x_0 + iy_0$. For $f'(z_0)$ to exist

(i) It is necessary that (5.6) holds.

(ii) It is sufficient that (5.6) holds and u, v are C^1 functions in some neighbourhood of (x_0, y_0)

(iii) If $f'(z_0)$ exists, then it can be computed by any one of

$$f' = u_x + iv_x = v_y - iu_y = u_x - iu_y = v_y + iv_x$$

<u>Example:</u> $f(z) = |z|^2 = z\bar{z} = x^2 + y^2 \implies u = x^2 + y^2,\ v = 0$

$$u_x = 2x,\ u_y = 2y,\ v_x = 0,\ v_y = 0$$

Thus $u_x = v_x$ and $u_y = -v_x$ gives $(x, y) = (0, 0)$. Hence $f'(z)$ exists only when $z = 0$ by (5.6) and it does exists with value $f'(0) = 0$ by (ii) and (iii) of the theorem

<u>Definition:</u> $f(z)$ is analytic at z_0 if it is differentiable at all points in a neighbourhood of z_0. If not, then it is singular at z_0. $f(z)$ is analytic in a domain D if it is analytic at every point of the domain.

<u>Example:</u>

(1) $f(z) = |z|^2$ is nowhere analytic (although it is differentiable at $z = 0$).

(2) $f(z) = \dfrac{1}{z}$ is analytic everywhere except at $z = 0$, which is a singular point of $f(z)$.

(3) $f(z)=\bar{z}$ is nowhere differentiable so that nowhere analytic.

(4) $f(z)=z^n (n \geq 0)$ is analytic everywhere, such function are called entire.

(5) $f(z)=e^z=e^x(\cos(y)+i\sin(y)) \implies u=e^x\cos(y),\ v=e^x\sin(y)$

$$u_x=e^x\cos(y),\ v_y=e^x\cos(y),\ u_y=-e^x\sin(y),\ v_x=e^x\sin(y)$$

Cauchy-Riemann equations hold everywhere. The value of $f'(z)$ is $f'(z)=u_x+iv_x=e^x\cos(y)+ie^x\sin(y)$

(6) $f(z)=\sin(z)=\sin(x)\cosh(y)+i\cos(x)\sinh(y)$

$$u_x=\cos(x)\cosh(y),\ v_y=\cos(x)\cosh(y),\ u_y=\sin(x)\sinh(y),\ v_x=-\sin(x)\sinh(y)$$

so that again, Cauchy-Riemann equations hold everywhere. Since $u,\ v$ are C^1 functions everywhere, we also have

$$f'(z)=u_x+iv_x=\cos(x)\cosh(y)+-i\sin(x)\sinh(y)$$

(7) $f(z)=e^{\sin(z)} \implies f(z)=e^{g(z)};\ g(z)=\sin(z)$ By Chain Rule

$$\frac{df}{dz}=\frac{de^g}{g}\frac{dg}{dz}=e^g\cos(z)=e^{\sin(z)}\cos(z)$$

Since both e^g and g are entire functions, so is their composition.

5.5.2 Cauchy-Riemann Equations in Polar Coordinates

Let $z=re^{i\theta}$ so that

$$f(z)=u(r,\theta)+iv(r,\theta)$$

for functions $u,\ v$ of independent variables $r,\ \theta$. Then

$$\begin{bmatrix} u_r \\ u_\theta \end{bmatrix}=\begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -r\sin(\theta) & r\cos(\theta) \end{bmatrix}\begin{bmatrix} u_x \\ u_y \end{bmatrix},\ \begin{bmatrix} v_r \\ v_\theta \end{bmatrix}=\begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -r\sin(\theta) & r\cos(\theta) \end{bmatrix}\begin{bmatrix} v_x \\ v_y \end{bmatrix}$$

which follow by $u_r=u_xx_r+u_yy_r,\ u_\theta=u_xx_\theta+u_yy_\theta$ and by $x=r\cos(\theta),\ y=r\sin(\theta)$. Similarly for v. It follows that by (5.6),

$$u_r=\cos(\theta)u_x+\sin(\theta)u_y=\cos(\theta)v_y-\sin(\theta)v_x=\frac{1}{r}v_\theta$$
$$v_r=\cos(\theta)v_x+\sin(\theta)v_y=-\cos(\theta)u_y+\sin(\theta)u_x=-\frac{1}{r}u_\theta$$

Moreover,

$$f'(z)=u_x+iv_x=e^{-i\theta}(u_r+iv_r)$$

which can be verified by substituting u_r and using (5.6) again.

Example: $f(z) = \log(z) = \ln(r) + i\theta,\ 0 < r < \infty,\ -\pi < \theta < \pi.$ is a differentiable function everywhere in its domain since $u(r,\theta) = \ln(r)$, $v(r,\theta) = \theta$, then

$$u_r = \frac{1}{r},\ v_\theta = 1,\ u_\theta = 0,\ v_r = 0$$

so that

$$u_r = \frac{1}{r} v_\theta \text{ and} v_r = -\frac{1}{r} u_\theta = 0$$

Thus, Cauchy-Riemann equations hold with $u, v \in C^1$. It follows that f is everywhere differentiable for the domain $r \neq 0$ and $\theta \in (-\pi, \pi)$. The derivative is given by

$$\frac{d}{dz}\log(z) = e^{-i\theta}(u_r + iv_r) = e^{-i\theta}\frac{1}{r} = \frac{1}{z}$$

Analytic functions have a remarkable property:

Fact: If f is analytic in a domain, then $f^{(k)}$ exists for all order $k \geq 1$. Moreover, if $f = u(x,y) + iv(x,y)$, then partial derivatives of f and they are continuous there

Consider $f(z) = u + iv$ and suppose it is analytic in D. Then, by Cauchy-Riemann equations $u_x = v_y$, $u_y = -v_x$ $\forall (x,y) \in D$, which implies, by the fact above, that

$$\begin{aligned} u_{xx} = v_{yx},\ u_{yy} = -v_{xy} &\implies u_{xx} + u_{yy} = 0, \\ v_{yy} = u_{xy},\ v_{xx} = -u_{yx} &\implies v_{xx} + v_{yy} = 0,\ \forall (x,y) \in D \end{aligned}$$

Thus, Laplace's equations $\nabla^2 u = u_{xx} + u_{yy} = 0$ and $\nabla^2 v = v_{xx} + v_{yy} = 0$ hold for functions u and v over D.

Note: The Laplace's equations in polar coordinates is

$$u_{\theta\theta} + r^2 u_{rr} + r u_r = 0, \text{ for } u(r,\theta)$$

A function $h(x,y)$ is called harmonic in a domain D in xy-plane, if it is C^2 in D and satisfies $\nabla^2 h = 0$ for every $(x,y) \in D$. Clearly, if $f = u + iv$ is analytic over D, then u and v are harmonic functions over D.

Definition: A 2D function v is a harmonic conjugate in D of u if they are both harmonic in D and the Cauchy-Riemann equations $u_x = v_y$, $u_y = -v_x$ hold over D.

Fact: If $u(x,y)$ is harmonic in the rectangle $R = \{(x,y) :\ x_1 \leq x \leq x_2,\ y_1 \leq y \leq y_2\}$, then there exists a harmonic conjugate v of u in $D = \{(x,y) :\ x_1 \leq x \leq x_2,\ y_1 \leq y \leq y_2\ \}$ such that $f = u + iv$ is analytic in D.

Example: $u = 3xy^2 - x^3$ is clearly C^2 and satisfies

$$u_x = 3y^2 - 3x^2,\ u_y = 6xy,\ u_{xx} = -6x,\ u_{yy} = 6x, \nabla^2 u = 0$$

Hence, it is harmonic in the whole plane. We construct a harmonic conjugate v: By Cauchy-Riemann equations $u_x = v_y = 3y^2 - 3x^2$ so that $v = y^3 - 3x^2y + A(x)$ for some $A(x)$. Also,

$$v_x = -6xy + A'(x) = -u_y = -6xy \implies A'(x) = 0 \implies A(x) = c$$

for some $c \in \mathbb{C}$. Hence, $v(x,y) = y^3 - 3x^2y + c$ for any $c \in \mathbb{C}$ is a harmonic conjugate of u. Note that

$$f(z) = u + iv = -z^3 + ic$$

is analytic in the whole plane, i.e., entire.

5.5.3 Complex Integral Calculus

A curve C in xy-plane is simple if it does not intersect itself. If only the end points of C is an intersection, then C is called a simple closed curve. It is called smooth if the tangent vector is well defined at every point of C and it varies continuously along C. A curve C is called piecewise smooth if it can be partitioned into a finite number of smooth segments. A contour is a curve, which is piecewise smooth and simple. Given a contour C with initial point $A = z_0$ and final point $B = z_n$, let us partition C into n smooth segments by choosing $n-1$ points in addition

$$A = z_0, z_1, \dots, z_{n-1}, z_n = B$$

Let Q_k be points on the segment from z_{k-1} to z_k for $k = 1, \dots, n$ and let

$$\Delta z_k = z_k - z_{k-1},\ k = 1, \dots, n$$

which is a vector of length $|\Delta z_k|$. Also let $|\Delta z| = \max_k |\Delta z_k|$. The contour integral of $f(z)$ along C is defined as

$$\int_C f(z)dz = \lim_{n\to\infty,\ |\Delta z|\to 0} \sum_{k-1}^{n} f(Q_k)\Delta z_k = \lim_{n\to\infty,\ |\Delta z|\to 0} J_n \tag{5.7}$$

whenever the limit exists.

Example: Let us compute the contour integral of $f(z) = z$ along an arbitrary contour C using his definition. We will calculate the limit in (5.7) by first choosing $Q_k = z_{k-1}$ and second by choosing $Q_k = z_k$

$$J_n^1 = \sum_{k=1}^{n} f(Q_k)\Delta z_k = \sum_{k=1}^{n} f(z_{k-1})\Delta z_k = \sum_{k=1}^{n} z_{k-1}\Delta z_k$$
$$J_n^2 = \sum_{k=1}^{n} f(Q_k)\Delta z_k = \sum_{k=1}^{n} f(z_k)\Delta z_k = \sum_{k=1}^{n} z_k\Delta z_k$$

so that

$$J_n^1 + J_n^2 = z_n^2 - z_0^2 = B^2 - A^2$$

Hence,

$$\lim_{|\Delta z|\to 0,\, n\to\infty} (J_n^1 + J_n^2) = B^2 - A^2 = 2\int_C f(z)dz$$

or

$$\int_C zdz = \frac{1}{2}(B^2 - A^2) = \frac{1}{2}(z_n^2 - z_0^2)$$

The following properties follow the definition easily

$$\int_C [\alpha f(z) + \beta g(z)]dz = \alpha \int_C f(z)dz + \beta \int_C g(z)dz$$

$$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz$$

$$\int_{-C} f(z)dz = -\int_C f(z)dz$$

where $\alpha, \beta \in \mathbb{C}$, $C_1 + C_2 = C$ (i.e., C_1 and C_2 joined is the contour C), and $-C$ is the contour C with direction reversed. It is also easy to see from the definition that if one writes $f(z) = u(x, y) + iv(x, y)$, $dz = dz + idy$, then

$$\begin{aligned}\int_C f(z)dz &= \int_C (u + iv)(dx + idy)\\ &= \int_C (udx - vdy) + i\int_C (udy + vdx)\end{aligned}$$

where the real and imaginary parts are "line integrals" in the xy-plane.

<u>**Example:**</u> Let us evaluate $\int_C z^2 dz$ with C as shown *insert image here*. We can parametrize C by $y = t$, $x = 4t^2$, t changes from 2 to -2. Then, since $(x + iy)^2 = x^2 - y^2 + i2xy$, we have $u = x^2 - y^2$, $v = 2xy$ and

$$\begin{aligned}\int_C z^2 dz &= \int_C [(x^2 - y^2)dx - 2xydy] + i\int_C [2xydx + (x^2 - y^2)dy]\\ &= \int_2^{-2} [[(4 - t^2)^2 - t^2](-2tdt) - 2t(4 - t^2)dt] + i\int_2^{-2} [2t(4 - t^2)(-2t)dt + [(4 - t^2)^2 - t^2]dt\\ &= \int_2^{-2} \underbrace{[(4 - t^2)^2 + 4 - 2t^2](-2t)}_{\text{odd}}\overset{0}{dt} + i\int_2^{-2} \underbrace{[(4 - t^2)(4 - 5t^2) - t^2]}_{\text{even}}\, dt\\ &= 2i\int_2^0 [5t^4 - 25t^2 + 16]dt\\ &= 2i\ [t^5 - \frac{25}{3}t^3 + 16t]\Big|_2^0 = i\frac{16}{3}\end{aligned}$$

<u>**Example:**</u> Let $f(z) = (z - a)^n$, $n \in \mathbb{Z}$, with C circular as shown *insert image here*. A parametrization of C is

$$C : z = a + Re^{i\phi},\ 0 \le \phi < 2\pi$$

Thus, $dz = Rie^{i\phi}d\phi$ and

$$\begin{aligned}
\int_C (z-a)^n dz &= \int_0^{2\pi} R^n e^{in\phi}(iRe^{i\phi})d\phi \\
&= \int_0^{2\pi} iR^{n+1}e^{i(n+1)\phi}d\phi \\
&= \begin{cases} \frac{R^{n+1}}{n+1}e^{i(n+1)\phi}\,|_0^{2\pi} & ,n \neq -1 \\ i2\pi & ,n = -1 \end{cases} \\
&= \begin{cases} 0 & ,n \neq -1 \\ i2\pi & ,n = -1 \end{cases}
\end{aligned}$$

Note that $\int_C zdz = 0$ for any closed contour, whereas $\int_C \frac{1}{z}dz = 2\pi i$. Similarly,

$$\int_C (z-a)^n dz = 0 \;\forall n \neq -1$$

where as $\int_C \frac{1}{z-a} = 2\pi i$

5.5.4 A Bound on The Contour Integral

$J_n = \sum_{k=1}^n f(Q_k)\Delta z_k$ so that

$$|J_n| = |\sum_{k=1}^n f(Q_k)\Delta z_k| \leq \sum_{k=1}^n |f(Q_k)||\Delta z_k|$$

and if there is $m > 0$ such that $|f(z)| \leq m$, $\forall z \in C$, then $|J_n| \leq m \displaystyle\sum_{k=1}^n |\Delta z_k|$. As $n \to \infty$ and $|\Delta z| \to 0$, we have

$$\lim_{n\to\infty,\,|\Delta z|\to 0} \sum_{k=1}^n |\Delta z_k| = \text{length of } C = L$$

Therefore

$$\left|\int_C f(z)dz\right| \leq M \cdot L \tag{5.8}$$

where M is an upper bound for $f(z)$, $z \in C$ and L is the length of the contour C.

Example:

$$I = \int_C \frac{\sin(z)}{z(z^2+9)}dz$$

where C is the positively oriented contour $|z| = 5$ (circle of radius 5 centered at origin). We have

$$\begin{aligned}
|\sin(z)| &= |\sin(x)\cosh(y) + i\cos(x)\sinh(y)| \\
&= [\sin^2(x)\cosh^2(y) + \cos^2(x)\sinh^2(y)]^{1/2} \\
&= [\sin^2(x) + \sinh^2(y)]^{1/2} \\
&\leq [1 + \sinh^2 y]^{1/2} = \cosh(y) \leq \cosh(5)
\end{aligned}$$

and, for $|z| = 5$ on C, two vectors shown have minimum length 2, so

$$|z^2 + 9| = |z - 3i||z + 3i| \geq 4$$

It follow that

$$\frac{|\sin(z)|}{|z||z^2 + 9|} \leq \frac{\cosh(5)}{20}$$

Also, $L = 2\pi 5 = 10\pi$, so that $|I| \leq \frac{\pi}{2}\cosh(5)$If C is a closed contour, then we will denote the integral along C of a function $f(z)$ by

$$\oint_C f(z)dz$$

Theorem: (Cauchy-Goursat) If $f(z)$ is analytic on and inside a closed contour C, then

$$\oint_C f(z)dz = 0$$

Cauchy-Goursat Theorem can be proved using the Cauchy-Euler equations on $f = u + iv$ and Stoke's Theorem.

Corollary 1: If f is analytic in a simply connected domain D (no holes), then $\int_C f(z)dz = 0$ for every simple closed contour C lying entirely in D.

Corollary 2: If f is analytic in a simply connected domain D, then $\int_C f(z)dz$ is independent of path in D, i.e., the value of the contour integral depends only end points of C.

To see why Corollary 2 is true, consider any two C_1, C_2 with same end point. Then, $C = C_2 - C_1$ is a (positively oriented) simple closed contour so that

$$\oint_C f(z)dz = \int_{C_2 - C_1} f(z)dz = \int_{C_2} f(z)dz - \int_{C_1} f(z)dz$$

by Cauchy-Goursat Theorem, which gives path independence.

Corollary 3: Consider two simple closed contours C_1 and C_2, both positively oriented, such that C_1 can be obtained from C_2 by a "continuous deformation". Then,

$$\oint_{C_1} f(z)dz = \oint_{C_2} f(z)dz$$

for any f which is analytic in a domain D that contains C_1 and C_2 both. To see why, consider that $C = C_3 + C_2 - C_3 - C_1$ is a positively oriented simple closed contour so that $\oint_C f(z)dz = 0$

Example: By Corollary 3 and the example on page 74, we have that

$$\oint_C (z-a)^n dz = \begin{cases} lr2\pi i & , n = -1 \\ 0 & , n \neq -1 \end{cases}$$

not only for a circle but for any contour C that encloses the point a in the positive direction.

Example:

$$I = \oint_C \frac{dz}{z^2(z-2)(z-4)}$$

We have that, by partial fraction expansion

$$\begin{aligned} I &= \oint_C \left[\frac{3}{32}\frac{1}{z} + \frac{1}{8}\frac{1}{z^2} - \frac{1}{8}\frac{1}{z-2} + \frac{1}{32}\frac{1}{z-4} \right] \\ &= \frac{3}{32}\oint_C \frac{1}{z}dz + \frac{1}{8}\oint_C \frac{1}{z^2} - \frac{1}{8}\oint_C \frac{1}{z-2} + \frac{1}{32}\oint_C \frac{1}{z-4}dz \\ &= \frac{3}{32}(2\pi i) + \frac{1}{8}(0) - \frac{1}{x}(2\pi i) + \frac{1}{32}(0) = -i\frac{\pi}{16} \end{aligned}$$

where the first three terms are by the example above and the last by Cauchy-Goursat Theorem.

5.5.5 Fundamental Theorem of Complex Calculus

Let f be analytic in a simply connected domain D and let z_0 be any point inside D. Then

i)

$$G(z) = \int_{z_0}^{z} f(\zeta)d\zeta$$

is analytic in D and, for every $z \in D$, $G'(z) = f(z)$, where the integral from z_0 to z is a contour integral that starts at z_0 and ends at z along any simple contour C lying in D.

ii) If $F(z)$ is any anti-derivative of $f(z)$, then

$$\int_{z_0}^{z} f(\zeta)d\zeta = F(z) - F(z_0) \tag{5.9}$$

Proof: The contour integral along C from z_0 to z is independent of path by Corollary 2. Let Δz be such that $z + \Delta z \in D$ and note that

$$\begin{aligned} \frac{G(z+\Delta z) - G(z)}{\Delta z} &= \frac{1}{\Delta z}\left[\int_{z_0}^{z+\Delta z} f(\zeta)d\zeta - \int_{z_0}^{z} f(\zeta)d\zeta \right] \\ &= \frac{1}{\Delta z}\int_{z}^{z+\Delta z} f(\zeta)d\zeta \\ &= \frac{1}{\Delta z}\int_{z}^{z+\Delta z} f(z)d\zeta + \frac{1}{\Delta z}\int_{z}^{z+\Delta z} [f(\zeta) - f(z)]d\zeta \\ &= f(z) + \frac{1}{\Delta z}\int_{z}^{z+\Delta z} [f(\zeta) - f(z)]d\zeta \end{aligned}$$

By continuity of f at z, given any $\epsilon > 0$, there exists $f > 0$ such that $|\Delta z| < \delta$ implies $|f(z) - f(\zeta)| < \epsilon$. Then we have that

$$\left| \frac{G(z+\Delta z) - G(z)}{\Delta z} - f(z) \right| < \frac{1}{|\Delta z|}\epsilon|\Delta z| \text{ for all } |\Delta z| < \delta$$

This proves (i). To see (ii), note by (i) that $G(z)$ is an anti-derivative of $f(z)$. If $F(z)$ is any other anti-derivative, then $G'(z) - F'(z) = 0$ so that $G(z) = F(z) + C$ for a constant C. Now, $G(z_0) = F(z_0) + V = 0$ so that $C = -F(z_0)$, i.e., $G(z) = F(z) - F(z_0)$ so that (5.9) holds.

Example:

$$\int_{2i}^{3} \sin(z)dz = -\cos(z)|_{2i}^{3} = -\cos(3) + \cos(2i) = \cosh(2) - \cos(3)$$

by the fact that $\sin(z)$ is entire (analytic in an open set includes the points 3 and $2i$) and its anti-derivative is $-\cos(z)$

Example:

$$\int_{1+i}^{-i} \frac{1}{z}dz$$

This problem is not well defined since $\frac{1}{z}$ is not analytic at the origin. The value of the integral differs if the path is C_1 or C_2.In fact,

$$\int_{C_1 - C_2} \frac{1}{z}dz = 2\pi i \implies \int_{C_1} \frac{1}{z}dz \neq \int_{C_2} \frac{1}{z}dz$$

In order to evaluate these two integrals using the Fundamental Theorem, we need to choose different anti-derivatives of $\frac{1}{z}$. For C_2, a suitable anti-derivative is $\log(z)(0 < r < \infty,\ -\pi < \theta < \pi)$. For C_1, it may be $log(z)(0 < r < \infty,\ 0 < \theta < 2\pi)$

$$\begin{aligned}
\int_{C_1} \frac{1}{z}dz &= \log(z)\ |_{1+i}^{-i} \\
&= i\frac{3\pi}{2} - (\ln(\sqrt{2}) + i\frac{\pi}{4}) \\
&= -\ln(\sqrt{2}) + i\frac{5\pi}{4} \\
\int_{C_2} \frac{1}{z}dz &= \log(z)\ |_{1+i}^{-i} \\
&= -i\frac{\pi}{2} - (\ln(\sqrt{2}) + i\frac{\pi}{4}) \\
&= -\ln(\sqrt{2}) - i\frac{3\pi}{4}
\end{aligned}$$

5.5.6 Cauchy Integral Formula

Let $F(z)$ be analytic in a simply connected domain D and let C be a simple (piecewise smooth) closed contour in D with positive orientation. If a point a is enclosed by C, then

$$\oint_C \frac{f(z)}{z-a}dz = i2\pi f(a) \tag{5.10}$$

Proof: By deformation corollary (Corollary 3), for the circle C' with center at a shown above *insert image here*, we have

$$\oint_C \frac{f(z)}{z-a}dz = \oint_{C'} \frac{f(a)}{z-a}dz + \oint_{C'} \frac{f(z)-f(a)}{z-a}dz$$
$$= 2\pi i f(a) + \underbrace{\oint_{C'} \frac{f(z)-f(a)}{z-a}dz}_{I}$$

By continuity of f at a, give $\epsilon > 0$, there is $f > 0$ such that $|z-a| < \delta$ implies $|f(z)-f(a)| < \epsilon$. Let ρ be less than δ so that for all $|z-a| = \rho$, we have $|f(z) - f(a)| < \epsilon$. This gives that

$$|I| = \left| \oint_{C'} \frac{f(z)-f(a)}{z-a}dz \right| < \frac{\epsilon}{\rho} 2\pi\rho = 2\pi\epsilon$$

implying, for any $\epsilon > 0$, that,

$$\left| \oint_C \frac{f(z)}{z-a}dz - 2\pi i f(a) \right| < 2\pi\epsilon$$

and hence (5.10) follows.

Example:

$$I = \oint_C f(z)dz,\ f(z) = \frac{e^z}{(z-2)(z+4)}$$

Since $g(z) = \dfrac{e^z}{z+4}$ is analytic on and inside $C : (|z| < 3)$, we can write $I = 2\pi i g(2) = i\dfrac{\pi e^2}{3}$ by Cauchy Integral Formula (5.9).

Generalized Cauchy Integral Formula:

(5.10) generalizes to

$$\oint_C \frac{f(z)}{(z-a)^{n+1}}dz = \frac{2\pi i}{n!} f^{(n)}(a) \ :\ n = 0, 1, 2, \ldots \tag{5.11}$$

Example: Let C be the positively oriented unit circle and note by (5.10) (with $n = 2$, $f(z) = e^z$, $a = 0$) that

$$\oint_C \frac{e^z}{z^3}dz = \frac{2\pi i}{2!} \frac{d^2}{dz^2} e^z \bigg|_{z=0} = \frac{2\pi i}{2} e^0 = \pi i$$

Formula (5.11) shows that if f is analytic inside and on the contour C, then the derivatives of every order of f exists at each point a inside C

Remark: (5.11) can be remembered easily as it is obtained by taking derivatives of (5.9) with respect to a on both sides.

5.6 Complex Series

Consider

$$\sum_{n=1}^{\infty} c_n = c_1 + c_2 + c_3 + \dots \ ; \ c_n \in \mathbb{C}$$

which is said to converge to a number S if $\forall \epsilon > 0$, $\exists$ and integer N such that the partial sums

$$S_n = c_1 + \cdots + c_n$$

satisfy $|S_n - S| < \epsilon \ \forall n > N$. Otherwise, it diverges.

Cauchy Convergence:

An infinite series $\sum_{n=1}^{\infty} c_n$converges if the partial sums S_n is a "Cauchy Sequence", i.e, if $\forall \epsilon > 0$, there is $N(\epsilon)$ such that $|S_m - S_n| < \epsilon \ \forall m, n > N(\epsilon)$

Comparison Test:

If $|c_n| \leq m_n$ for all n greater than some integer N_0 and if $\sum_{n=N_0}^{\infty} m_n$ converges, then $\sum_{n=1}^{\infty} c_n$ also converges

Ratio Test:

If $\lim_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right| = L$, then $\sum_{n=1}^{\infty} c_n$ converges if $L < 1$, diverges if $L > 1$, and the test of L is inconclusive if $L = 1$

A Necessary Condition:

[1] If $\sum_{n=1}^{\infty} c_n$ converges, then $\lim_{n\to\infty} c_n = 0$

Example:

(1) $\sum_{n=0}^{\infty} \frac{(1+i)^n}{n!}$ converges since $\left|\frac{(1+i)^n}{n!}\right| = \frac{\sqrt{2}^n}{n!} < \frac{\sqrt{2}^n}{2^n}$ for $n \geq 4 = N_0$ and since $\sum_{n=0}^{\infty} \left(\frac{\sqrt{2}}{2}\right)^n$ converges.

(2) $\sum_{n=2}^{\infty} e^{-(2+3i)n}$ converges by the ratio test since

$$\lim_{n\to\infty} \left(\frac{c_{n+1}}{c_n}\right) = \lim_{n\to\infty} |e^{-(2+3i)}| = e^{-2} = L < 1$$

[1]The necessary condition follows by Cauchy Convergence by setting $m = n - 1$

(3) $\displaystyle\sum_{n=0}^{\infty}\left(\frac{1+n}{2+n}\right)^3$ diverges since the necessary condition fails, i.e., since

$$\lim_{n\to\infty}\left(\frac{1+n}{2+n}\right)=1\neq 0$$

5.6.1 Power Series

$$\sum_{n=0}^{\infty}a_n(z-a)^n;\ a\in\mathbb{C},\ a_n\in\mathbb{C},\ n=0,1,2,\ldots$$

and z takes the complex plane as well.

Convergence Test for Power Series

If $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=L$, then the power series converges in the disk $|z-a|<\dfrac{1}{L}$, diverges in $|z-a|>\dfrac{1}{L}$. If $L=\infty$, then the power series only converges at $z=a$ and if $L=0$, then it converges in the whole complex plane, i.e., $\forall|z-a|<\infty$.

Proof: Let

$$p_n=\left|\frac{a_{n+1}(z-a)^{n+1}}{a_n(z-a)^n}\right|=\left|\frac{a_{n+1}}{a_n}\right||z-a|$$

Then, by the ratio test applied to $c_n=a_n(z-a)^n$, we have that

$$\begin{aligned}|z-a|<\frac{1}{L}&\implies\lim_{n\to\infty}p_n<1\implies\text{convergent}\\|z-a|>\frac{1}{L}&\implies\lim_{n\to\infty}p_n>1\implies\text{divergnet}\end{aligned}$$

Example: $\displaystyle\sum_{n=0}^{\infty}z^n$ converges if and only if $|z|<1$ since $L=1$ in the above test and by the fact that when $|z|=1$, $\displaystyle\lim_{n\to\infty}|z^n|\neq 0$ and the necessary condition for convergence fails.

Fact: If a power series is convergent at a point $z_0\in\mathbb{C}$, then it convergence $\forall z$ in the open disk $|z-a|<|z_0-a|$

Proof: By convergence at z_0, we have $\displaystyle\lim_{n\to\infty}a_n(z_0-a)^n=0$, by the necessary condition for convergence. Hence, there is $K>0$ such that $\forall n$, it holds that

$$|a_n(z_0-a)^n|<K,\ \text{or,}\ |a_n|<\frac{K}{|z_0-a|^n}$$

This implies

$$|a_n(z-a)^n|<K\left|\frac{z-a}{z_0-a}\right|^n$$

for z such that $\left|\frac{z-a}{z_0-a}\right|^n < 1$, $\sum_{n=0}^{\infty}\left|\frac{z-a}{z_0-a}\right|^n$ is convergent. By comparison test, $\sum_{n=0}^{\infty} a_n(z-a)^n$ is also convergent in the disk $|z-a| < |z_0-a|$. By this fact, there are only three possibilities for the convergence of a power series:

1) The power series converges only at $z = a$. (trivial case)

2) It converges everywhere, i.e., for all z such that $|z-a| < \infty$.

3) There is $R > 0$ such that it converges in the disk $|z-a| < R$ and diverges in $|z-a| > R$. Such an $R > 0$ is called the radius of convergence of the power series. In 1 and 2, we may say $R = 0$ and $R = \infty$, respectively.

5.6.2 Taylor Series

If f is analytic at $a \in \mathbb{C}$, then, by definition of analyticity, there is a disk $|z-a| < \rho$ in which $f'(z)$ exists everywhere. By generalized Cauchy Integral Formula, derivatives of all order f also exist in this disk.

Theorem:[2](Taylor Series) Let f be analytic in a domain D with $a \in D$. Let R be such that the disk $|z-a| < R$ lies in D. Then, for all z in the disk, we have

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(z-a)^n \tag{5.12}$$

i.e, the power series on the right converges to the value at z of f. This representation of $f(z)$ is unique. Note by the explanation before the statement of the Theorem that if f is analytic at a point a, then a Taylor Series Expansion (5.12) exists in a disk $|z-a| < \rho$ (although ρ may be small number). The uniqueness statement implies, that if we obtain a power series for $f(z)$, i.e., find a_n such that $f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n$, then $a_n = \frac{f^{(n)}(a)}{n!}$ $\forall n \geq 0$, necessarily.

Example

(1)

$$\frac{1}{1-z} = 1 + z + z^2 + \cdots + z^k + \cdots = \sum_{k=0}^{\infty} z^k$$

converges for $|z| < 1$, i.e., in the disk of radius one centred at the origin. Hence, this power series about $a = 0$ is also the Taylor Series of $(1-z)^{-1}$ by uniqueness, i.e.,

$$\frac{1}{n!}\frac{d^n}{dz^n}\left(\frac{1}{1-z}\right)\bigg|_{z=0} = 1,\ \forall n \geq 0$$

(2)

$$\frac{1}{1+z^2} = 1 - z^2 + z^4 - z^6 + \cdots = \sum_{n=0}^{\infty}(-1)^n z^{2n}$$

[2]Proof of the Taylor Series result is done using generalized Cauchy Integral Formula(See p. 1215 in Section 24.2 of the textbook)

converges if and only if $|z| < 1$ as well since $|-z^2| < 1$ if and only if $|z| < 1$. This is the Taylor Series about $a = 0$.

(3)

$$f(z) = \frac{e^z}{\cos(z)}$$

Its Taylor Series about $a = 0$ can be determined from

$$1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots = (a_0 + a_1 z + a_2 z^2 + \dots)(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \dots)$$

which gives $a_0 = 1,\ a_1 = 1,\ a_2 = 1,\ a_3 = \frac{2}{3}$, etc. Since $f(z)$ is analytic in $|z| < \pi/2$, the series converges for $|z| < \pi/2$, and only for such z. That is, the radius of convergence of $f(z)$ is $\pi/2$.

The past example uses the multiplication property of power series. If $\alpha, \beta \in \mathbb{C}$, then

$$\alpha \sum_{n=0}^{\infty} a_n (z-a)^n + \beta \sum_{n=0}^{\infty} b_n (z-a)^n = \sum_{n=0}^{\infty} (\alpha a_n + \beta b_n)(z-a)^n$$

$$\left(\sum_{n=0}^{\infty} a_n (z-a)^n\right) \left(\sum_{n=0}^{\infty} b_n (z-a)^n\right) = \sum_{n=0}^{\infty} \left(\sum_{j=0}^{n} a_j b_{n-j}\right) (z-a)^n$$

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n \implies f'(z) = \sum_{n=0}^{\infty} n a_n (z-a)^{n-1}$$

where $f'(z)$ and $f(z)$ have the same radius of convergence

Example

(1)

$$\frac{e^z}{1-z} = \left(\sum_{n=0}^{\infty} z^n\right) \left(\sum_{n=0}^{\infty} \frac{z^n}{n!}\right) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{1}{k!}\right) z^n$$

(2)

$$\begin{aligned} \frac{d}{dz} e^z &= \frac{d}{dz} \sum_{n=0}^{\infty} \frac{z^n}{n!} \\ &= \sum_{n=1}^{\infty} \left(\frac{n}{n!}\right) z^{n-1} \\ &= \sum_{n=1}^{\infty} \frac{1}{(n-1)!} z^{n-1} \end{aligned}$$

5.6.3 Laurent Series

A function like $\dfrac{\sin(z)}{z^2}$ has no Taylor Series Expansion about $a = 0$ since it is not analytic at $z = 0$. However, it can still be expanded in power of z:

$$\frac{\sin(z)}{z^2} = \frac{1}{z} - \frac{1}{3!}z + \frac{1}{5!}z^3 - \frac{1}{7!}z^5 + \dots$$

which differ from a power series by the existence of negative powers of z ($\frac{1}{z}$ above). Such a series is called a Laurent Series.

Theorem: Let D be a closed region between and including two concentric circles C_i and C_0, centred at a. If $f(z)$ is analytic in the interior of D, then it has the Laurent Series Expansion

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n \tag{5.13}$$

valid inside D, where c_n $(n \in \mathbb{Z})$ are uniquely given by

$$c_n = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{(\zeta - a)^{n+1}} d\zeta$$

for any positively oriented, simple closed contour C lying inside D and enclosing a.

Remark: If $f(z)$ is also analytic on and inside C_i, then for $n \geq 0$, the generalized Cauchy Integral Formula (5.10) gives $c_n = \dfrac{f^{(n)}(a)}{n!}$ and for $n < 0$, Cauchy-Goursat Theorem gives $c_n = 0$. This reduces (5.13) to a Taylor Series (5.12).

Example: Let $f(z) = \dfrac{1}{z+i}$ and consider $a = 0$. Consider the regions D_1 and D_2, in both of which $f(z)$ has a Taylor Series about $a = 0$ valid in D_1 given by

$$D_1(|z| < 1) : \ \frac{1}{z+i} = \frac{1}{i}\frac{1}{1-iz} = -i\sum_{n=0}^{\infty}(iz)^n$$

In D_2, we are in the situation of the Laurent Series Theorem, where C_i : unit circle and C_0 : circle of radius ∞. Hence

$$D_2(|z| > 1) : \ \frac{1}{z+i} = \frac{1}{z}\left(\frac{1}{1+\frac{i}{z}}\right) = \frac{1}{z}\sum_{n=0}^{\infty}\left(-\frac{i}{z}\right)^n$$

is the Laurent series about $a = 0$ valid in $|z| > 1$, i.e., D_2. Suppose now that $a = 2$. Then, there are again two regions inside which $f(z)$ is analytic: $D_3 : (|z-2| < \sqrt{5})$ and $D_4 : (|z-2| > \sqrt{5})$. In both regions, we are again in the framework of Taylor and Laurent

Series Theorem.

$$
\begin{aligned}
D_3: \ \frac{1}{z+1} &= \frac{1}{z-2+2+i} \\
&= \frac{1}{2+i}\frac{1}{1+\frac{z-2}{2+1}} \\
&= \frac{1}{2+i}\sum_{n=0}^{\infty}\left(-\frac{z-2}{2+i}\right)^n \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2+i)^{n+1}}(z-2)^n \text{ (Taylor Series about } a=2)
\end{aligned}
$$

$$
\begin{aligned}
D_4: \ \frac{1}{z+i} &= \frac{1}{z-2+2+i} \\
&= \frac{1}{z-2}\frac{1}{1+\frac{2+i}{z-2}} \\
&= \frac{1}{z-2}\sum_{n=0}^{\infty}(-1)^n\frac{(2+i)^n}{z-2}^n \\
&= \sum_{n=0}^{\infty}(-1)^n(2+i)^n(z-2)^{-1-n} \\
&= \sum_{k=-\infty}^{-1}(-1)^{k+1}(2+i)^{-(k+1)}(z-2)^k \text{ (Laurent Series about } a=2)
\end{aligned}
$$

Note that the series expansion in D_4 after the third equality is valid (convergent) of and only if $\left|\frac{2+i}{z-2}\right| < 1$, if and only if $|z-2| > \sqrt{5}$

Example: $f(z) = \frac{1}{\sin(z)}$ has singularities at $k\pi$ for $k \in \mathbb{Z}$. If $a = \pi$, then $f(z)$ has a Laurent series valid in $D : (0 < |z-\pi| < \pi)$ shown, The series expansion can be obtained as follows

$$\frac{1}{\sin(z)} = \frac{1}{\sin(z-\pi+\pi)} = -\frac{1}{\sin(z-\pi)} = -\frac{1}{z-\pi}\frac{z-\pi}{\sin(z-\pi)}$$

Let $w = z - \pi$ and note that

$$\frac{w}{\sin(w)} = \frac{w}{\sum\limits_{k \text{ odd}}\frac{w^k}{k!}} = \frac{1}{1-\frac{w^2}{3!}+\frac{w^4}{5!}-\dots} = 1+\frac{1}{6}w^2+\frac{7}{360}w^4+\dots$$

where the last equality is obtained by long division. The series thus obtained is convergent for all $|w| < \pi$ and hence for all $|z-\pi| < \pi$. The Laurent Series

$$\frac{1}{\sin(z)} = -\frac{1}{z-\pi}\left[1+\frac{1}{6}(z-\pi)^2+\frac{7}{360}(z-\pi)^4+\dots\right] = -\frac{1}{z-\pi}-\frac{1}{6}(z-\pi)-\dots$$

is, on the other hand, valid for all z such that $0 < |z-\pi| < \pi$

Example: $e^{1/z}$ is analytic in $0 < |z| < \infty$. Its Laurent Series about 0 is obtained from $e^w = \displaystyle\sum_{k=0}^{\infty} \frac{w^k}{k!}$ with $w = 1/z$

$$e^{1/z} = \sum_{k=0}^{\infty} \frac{1}{k!}(z^{-1})^k = \sum_{n=-\infty}^{0} \frac{1}{(-n)!} z^n$$

5.7 Classification of Singularities

If $f(z)$ is not analytic at a point $z = a$ but analytic in a neighbourhood of it, then a is an isolated singular point of f. Otherwise, it is a non-isolated singular point.

Example

(1)
$$f(z) = \frac{1}{\sin(\frac{1}{z})}$$
is singular at $z = 0$ and at $z = \dfrac{1}{k\pi}$ for $k \neq 0,\ k \in \mathbb{Z}$. All singularities at $\dfrac{1}{k\pi}$ are isolated but $z = 0$ is a non-isolated singular point since every disk $0 < |z| < \rho$ contains infinitely many singularities $\dfrac{1}{k\pi}$ for sufficiently large $k \in \mathbb{Z}$, no matter how small $\rho > 0$ is.

(2) $f(z) = \log(z)$ is singular at all points on its branch-cut, all singularities on which are non-isolated.

(3) $f(z) = |z|^2$ was shown to be nowhere analytic so that each point on the complex plane is a non-isolated singularity.

Suppose now that f has an isolated singularity at a. This means that, there is $\rho > 0$, such that f is analytic in the deleted neighbourhood $0 < |z - a| < \rho$ of a. Hence, f has a Laurent Series Expansion about a:

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n = \cdots + c_{-2}\frac{1}{(z-a)^2} + c_{-1}\frac{1}{z-a} + c_0 + c_1 z - a + \ldots$$

If there are only a finite number N of negative indexed terms in this expansion, then a is a pole of order N of f:

$$f(z) = c_{-N}\frac{1}{(z-a)^N} + \cdots + c_{-1}\frac{1}{z-a} + c_0 + c_1(z-a) + \ldots$$

If, there are infinitely many negative indexed terms, then a is an essential singularity of f.

Example

(1) $f(z) = e^{1/z} = \displaystyle\sum_{n=-\infty}^{0} \frac{1}{(-n)!} z^n$ has an essential singularity at $z = 0$.

(2) $f(z) = \dfrac{1}{z^2(1-z)}$ has two Laurent series about $z = 0$:

$$\begin{aligned} 0 < |z| < 1: \ f(z) &= \frac{1}{z^2}(1 + z + z^2 + \dots) \\ |z| > 1: \ f(z) &= -\frac{1}{z^3}\left(\frac{1}{z} + \frac{1}{z^2} + \dots\right) \end{aligned}$$

According to our classification, there is a pole of order 2 of f, and 0 is not an essential singularity.

5.7.1 Picard's Theorem

If $z = a$ is an essential singularity of $f(z)$, then in each neighbourhood of a, $f(z) = c$ for every given complex number c, except at most one number for an infinite number of values of z.

Example: $e^{1/z} = c \implies \dfrac{1}{z} = \log(c) = \ln|c| + i(\theta_0 + 2\pi k)$ so that $z_k = \dfrac{1}{\ln|c| + i(\theta_0 + 2\pi k)}$ for $k \in \mathbb{Z}$. The exception to c values here is $c = 0$!!

If $z = a$ is a pole, then $|f(z)| \xrightarrow[z \to a]{} \infty$, because, say for a second order pole,

$$\begin{aligned} f(z) &= \frac{c_{-2}}{(z-a)^2} + \frac{c_{-1}}{z-a} + \underbrace{c_0 + c_1(z-a) + \dots}_{g(z)} \\ \implies &|(z-a)^2 f(z)| = |c_{-2} + c_{-1}(z-a) + g(z)(z-a)^2| \\ \lim_{z \to a} &\left\{|z-a|^2 |f(z)|\right\} = 0 \cdot \lim_{z \to a} |c_{-2}| \end{aligned}$$

so that $\lim_{z \to a} |f(z)|$ must be ∞.

If $z = a$ is an essential singularity, $|f(z)| \xrightarrow[z \to a]{} \infty$ is not true.

Example:

$$\begin{aligned} e^{-\frac{1}{z^2}} &\xrightarrow[\substack{x=0 \\ y \to 0}]{} \lim_{y \to 0} e^{\frac{1}{y^2}} = \infty \text{ but} \\ e^{-\frac{1}{z^2}} &\xrightarrow[\substack{y=0 \\ x \to 0}]{} \lim_{x \to 0} e^{-\frac{1}{x^2}} = 0 \end{aligned}$$

5.7.2 Relation to Zeroes

If f is analytic at a, then it has a Taylor series about a

$$f(z) = f(a) + f'(a)(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \dots$$

If $f(a) = 0,\ f'(a) = 0, \ldots, f^{(k-1)}(a) = 0,\ f^{(k)}(a) \neq 0$, then f is said to have a zero of order (or multiplicity) k at a. Then, the Taylor series of f is

$$f(z) = \sum_{n=k}^{\infty} \frac{f^{(n)}(a)}{n!}(z-a)^n = \frac{f^{(k)}(a)}{k!}(z-a)^k + \cdots + \left(\substack{\text{higher order} \\ \text{terms}}\right)$$

Example: $\sin(z)$ has a first order (simple) zero at every $z = k\pi$ ($k \in \mathbb{Z}$). $1 - \cos(z)$ has a 2^{nd} order zero at $z = 0$ and simple zeros at $2k\pi$. $\sin^3(z)$ has a 3^{rd} order zero at $z = 0$ and at $z = k\pi,\ k \neq 0$. $\sin(z^3)$ also has 3^{rd} order zero at $z = 0$.

Fact: If $p(z),\ q(z)$ have zeros of orders $P,\ Q$, respectively, at a, then $f(z) = \dfrac{p(z)}{q(z)}$ has a pole of order $Q - P$ at a provided $Q > P$ and is analytic at a (with a zero of order $P - Q$) of $Q \leq P$.

Example: $f(z) = \dfrac{(\pi - z)(z^4 - 3z^2)}{\sin^2(z)}$ has candidate singular points at $z = n\pi,\ n \in \mathbb{Z}$. Let $p(z) = (\pi - z)z^2(z^2 - 3),\ q(z) = \sin^2(z)$

$\underline{z = 0:}$ $P = 2,\ Q = 2 \implies$ analytic if $z = 0$

$\underline{z = \pi:}$ $P = 1,\ Q = 2$ since $\sin^2 z = \sin^2(z - \pi + \pi) = \sin^2(z - \pi) = (z - \pi)^2 - \dfrac{1}{3}(z - \pi)^4 + \ldots$

so that f has a simple pole at π.

$\underline{z = n\pi,\ n \neq 0,\ n \neq 1:}$ $P = 0,\ Q = 2 \implies 2^{nd}$ order poles at such points.

5.7.3 Residue Theorem

Let z_j be an isolated singularity of $f(z)$ and let c_j be a positively oriented circle of radius $\rho > 0$ about z_j such that f is analytic in the disk $0 < |z - z_j| \leq \rho$, (i.e., everywhere on and inside the circle except at z_j). The residue of f at z_j is

$$\operatorname*{Res}_{z=z_j} f(z) = \frac{1}{2\pi i} \oint_{C_j} f(z)dz$$

By the Laurent series of f in the disk $0 < |z - z_j| < \rho$ about z_j, it follows that the coefficient c_{-1} of $(z - z_j)^{-1}$ in the series is

$$c_{-1} = \operatorname*{Res}_{z=z_j} f(z)$$

Theorem: Let f be analytic on and inside a positively oriented contour C except at the isolated singularities $z_1, \ldots, z_k$ inside C. Then

$$\oint_C f(z)dz = 2\pi i \sum_{j=1}^{k} \operatorname*{Res}_{z=z_j} f(z)$$

Proof: By deformation results Corollary 3 on page 75, we have (in the figure above)

$$\oint_C f(z)dz = \oint_{C_1} f(z)dz + \oint_{C_2} f(z)dz = 2\pi i(\operatorname*{Res}_{z=z_1} f(z) + \operatorname*{Res}_{z=z_2} f(z))$$

Example:

$$I = \oint_C \underbrace{\frac{1}{(z-1)(z-2)^2}}_{f(z)} dz = 2\pi i \left[\operatorname*{Res}_{z=1} f(z) + \operatorname*{Res}_{z=2} f(z)\right]$$

$$\begin{aligned}
\operatorname*{Res}_{z=1} f(z) : \frac{1}{z-1}\frac{1}{(z-2)^2} &= \frac{1}{z-1}\frac{1}{(z-1-1)^2} \\
&= \frac{1}{z-1}[1+(z-1)+(z-1)^2+\dots]^2 \\
&\implies \operatorname*{Res}_{z=1} = 1 \text{ (coefficient of } \frac{1}{z-1} \text{ above)} \\
\operatorname*{Res}_{z=2} f(z) : \frac{1}{(z-2)^2}\frac{1}{z-2+1} &= \frac{1}{(z-2)^2}[1-(z-2)+(z-2)^2+\dots] \\
&\implies \operatorname*{Res}_{z=2} = -1 \text{ (coefficient of } \frac{1}{z-2} \text{ above)}
\end{aligned}$$

Hence, $I = 1 - 1 = 0$

Example: $I = \oint_C z^4 \sin\left(\frac{1}{z}\right) dz$; C is positive unit circle. There is only one singularity at $z = 0$ inside C, which is an essential singularity. The Laurent series about 0 is

$$\begin{aligned}
f(z) = z^4 \sin(\frac{1}{z}) &= z^4 \left(\frac{1}{z} - \frac{1}{3!}\frac{1}{z^3} + \frac{1}{5!}\frac{1}{z^5} - \frac{1}{7!}\frac{1}{z^7} + \dots\right) \\
&= z^3 - \frac{1}{3!}z + \frac{1}{5!}\frac{1}{z} - \frac{1}{7!}\frac{1}{z^3} + \dots \ , \ 0 < |z| < \infty
\end{aligned}$$

so that $\operatorname*{Res}_{z=0} f(z) = \frac{1}{5!} = \frac{1}{120}$. Hence, $I = 2\pi i \frac{1}{120} = \frac{i\pi}{60}$. This example makes the point that the residue is valid whenever the Lauren Series exists.

Calculation of Residue for Poles

Given that $f(z)$ has a a pole of order N at a, we have

$$\begin{aligned}
f(z) &= c_{-N}\frac{1}{(z-a)^N} + c_{-N+1}\frac{1}{(z-a)^{N-1}} + \dots \\
&\implies (z-a)^N f(z) = c_{-N} + c_{-N+1}(z-a) + \dots
\end{aligned}$$

so that $(z-a)^N f(z)$ has the Taylor series about a given by the right hand side. Hence,

$$\begin{aligned}
c_{-N+l} &= \frac{1}{l!}\frac{d^l}{dz^l}\left[(z-a)^N f(z)\right]\Big|_{z=a} \ ; \ l = 0, 1, 2, \dots, N-l, \dots \\
\implies c_{-1} &= \frac{1}{(N-1)!}\frac{d^{N-1}}{dz^{N-1}}\left[(z-a)^N f(z)\right]_{z=a} = \operatorname*{Res}_{z=a} f(z)
\end{aligned} \tag{5.14}$$

Application of Residues to Real Integrals

Consider the improper integral

$$I = \lim_{R\to\infty} \int_{-R}^{R} \frac{dx}{x^2+1} = \int_{-\infty}^{\infty} \frac{dx}{x^2+1}$$

Let $C := C_0 + C_R$ and consider the integral

$$J = \oint_C \frac{dz}{z^2+1};\ f(z) = \frac{1}{z^2+1}$$

If $R > 1$, then by the Residue Theorem, $J = 2\pi i \operatorname*{Res}_{z=1} f(z)$, as $z = i$ is the singularity inside C. Thus, $J = 2\pi i \left.\dfrac{1}{z+i}\right|_{z=i} = \pi$, by (5.14). On the other hand,

$$J = \int_{C_0} + \int_{C_R} = \int_{-R}^{R} \frac{dx}{x^2+1} + \int_{C_R} \frac{dz}{z^2+1}$$

so that

$$I = \lim_{R\to\infty} \int_{-R}^{R} \frac{dx}{x^2+1} = \lim_{R\to\infty} J - \lim_{R\to\infty} \int_{C_R} = \pi - \lim_{R\to\infty} \int_{C_R} f(z)dz$$

But for $z \in C_R$, we have, by the figure above *insert figure*, $|z-i| \leq R-1,\ |z+i| \leq \sqrt{R^2+1}$

$$|f(z)| = \left|\frac{1}{z^2+1}\right| = \frac{1}{|z-i||z+i|} \leq \frac{1}{(R-1)\sqrt{R^2+1}}$$

which gives that

$$\lim_{R\to\infty} \left|\int_{C_R} f(z)dz\right| \leq \lim_{R\to\infty} \frac{2\pi R}{(R-1)\sqrt{R^2+1}} = 0 \tag{5.15}$$

It follows that $\displaystyle\lim_{R\to\infty} \int_{C_R} = 0$ also and, hence $I = \pi$. Alternatively, (5.15) can also be shown noting that $z = Re^{i\theta}$, we have

$$\left|\int_{C_R} \frac{dz}{z^2+1}\right| = \left|\int_0^p i\frac{iRe^{i\theta}d\theta}{R^2e^{i2\theta}+1}\right| \geq \int_0^\pi \frac{R}{R^2+1}d\theta = \frac{\pi R}{R^2-1}$$

which gives the same result.

__**Example:**__ $I = \displaystyle\int_0^\infty \frac{\cos(ax)}{x^2+1}dx,\ a > 0$. This time consider

$$f(z) = \frac{e^{iaz}}{z^2+1},\ J = \int_C f(z)dz$$

where C is the same semi-circle as in the *figure* on page 89. We have

$$J = \int_{-R}^{R} \frac{e^{iax}}{x^2+1}dx + \int_{C_R} \frac{e^{iaz}}{z^2+1}dz \tag{5.16}$$

For any $R > 1,\ J = 2\pi i \operatorname{Res} f(z) = 2\pi i \left(\frac{e^{iaz}}{z+i}\right)\Bigg|_{z=i} = \pi e^{-a}.$

Further, for any $z \in C_R$, we have $z = Re^{i\theta}$ and

$$|f(z)| = \frac{|e^{iaz}|}{|z^2+1|} = \frac{|e^{ia(x+iy)}|}{|T^2e^{i2\theta}+1| \le \frac{e^{-ay}}{R^2+1};\ x+iy \in C_R}$$

Hence,

$$\lim_{R\to\infty}\left|\int_{C_R} f(z)dz\right| \le \lim_{\substack{R\to\infty\\ y\to\infty}} \frac{\pi Re^{-ay}}{R^2-1} = 0$$

Taking limits as $|z| \to \infty$ or $R \to \infty$ and $y \to \infty$ in (5.16), we get

$$\begin{aligned}\pi e^{-a} &= \int_{-\infty}^{\infty} \frac{e^{iax}}{x^2+1} + \lim_{R\to\infty}\int_{C_R} \frac{e^{iaz}}{z^2+1}dz \\ &= \int_{-\infty}^{\infty} \frac{e^{iax}}{x^2+1}dx\end{aligned}$$

Taking the real and imaginary parts of the two sides, we get

$$\int_{-\infty}^{\infty} \frac{\cos(ax)}{x^2+1}dx = \pi e^{-a},\ \int_{-\infty}^{\infty} \frac{\sin(ax)}{x^2+1}dx = 0$$

It follows, by the fact that the integrand $\dfrac{\cos(ax)}{x^2+1}$ is even:

$$I = \int_0^{\infty} \frac{\cos(ax)}{x^2+1}dx = \frac{1}{2}\pi e^{-a}$$

5.7.4 Definite Integrals

Example:

$$I = \int_0^{2\pi} \frac{1}{2-\sin(\theta)}d\theta = \oint_C \frac{\frac{dz}{iz}}{2 - \frac{z-z^{-1}}{2i}}$$

where $z = e^{i\theta}$ for a point on the positively oriented unit circle, so that

$$\cos(\theta) = \frac{z+z^{-1}}{2},\ \sin(\theta) = \frac{z-z^{-1}}{2i},\ dz = ie^{i\theta}d\theta = izd\theta$$

Hence,

$$I = \oint_C \frac{dz}{2iz - \dfrac{z^2-1}{2}} = \oint_C \overbrace{\frac{-2}{z^2-4iz-1}}^{f(z)} dz$$

Singularities of $f(z)$ are at $z_1 = (2+\sqrt{3})i,\ z_2 = (2-\sqrt{3})i$ and z_1 is outside the unit circle, whereas z_2 is inside. thus

$$I = 2\pi i \operatorname*{Res}_{z=z_2} f(z) = -2(2\pi i)\left|\frac{1}{z-z_1}\right|_{z=z_2} = -4\pi i\frac{1}{z_2-z_1} = \frac{2\pi}{\sqrt{3}}$$

Example:

$$I = \int_0^{\pi} \frac{\cos(t)}{1 - 2a\cos(t) + a^2} dt, \ -1 < a < 1$$

Since the integrand is an even function of t, we have

$$\begin{aligned} I &= \frac{1}{2} \int_{-\pi}^{\pi} \frac{\cos(t)}{1 - 2a\cos(t) + a^2} dt \\ &= \frac{1}{2} \oint_C \frac{(z + z^{-1})/2}{1 - 2a[(z + z^{-1})/2] + a^2} \frac{dz}{iz} \\ &= -\frac{1}{4ai} \oint_C \underbrace{\frac{z^2 + 1}{z\left(z^2 - \frac{1+a^2}{a} + 1\right)}}_{f(z)} dz \end{aligned}$$

The singularities of $f(z)$ are at $z_1 = 0, \ z_2 = a, \ z_3 = \frac{1}{a}$ and $z_1, \ z_2$ are inside z_3 is outside the unit circle C. Hence

$$I = \frac{2\pi i}{4ai} \left[\operatorname*{Res}_{z=0} f(z) + \operatorname*{Res}_{z=a} f(z)\right] = \left[\left.\frac{z^2 + 1}{z^2 - \frac{1+a^2}{a} z + 1}\right|_{z=0} + \left.\frac{z^2 + 1}{z(z - \frac{1}{a})}\right|_{z=a} \right] (-\frac{\pi}{2a})$$

$$\implies I = -\frac{\pi}{2a}\left(1 + \frac{a^2 + 1}{a^2 - 1}\right) = \frac{\pi a}{1 - a^2}$$