MATH 241 Engineering Mathematics I

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Chapter 1

Complex Numbers and Complex Algebra

1.1 Introduction

Complex numbers have been introduced to have complete set of solutions for an polynomial equation.

Notation: $z \in \mathbb{C}, z = a + ib$
a : the real part of z
b : the imaginary part of z
i : root to the $x^2 + 1 = 0$
$a \equiv Re\{a + ib\}$ $b \equiv Im\{a + ib\}$
Definitions:

1. If $z = Re\{z\}$, z is called "purely real"
2. If $z = Im\{z\}$, z is called "purely imaginary"
3. If $z_1 = z_2$, then $Re\{z_1\} = Re\{z_2\}$ and $Im\{z_1\} = Im\{z_2\}$

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1.2 Algebraic Definitions

1.2.1 Summation

Let $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$, then

$$z = z_1 + z_2 = a + ib$$

Where

$$a = a_1 + a_2$$

$$b = b_1 + b_2$$

Properties of Complex Summation:

(i) $z_1 + z_2 = z_2 + z_1$; Commutative

(ii) $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$; Associative

(iii) $z_1 + (0 + i0) = z_1$

(iv) for any $z_1 \in \mathbb{C}$, there exists a z_2 such that $z_1 + z_2 = 0$

(v) Additive inverse of $z = a + ib$ is $-z = -a - ib$ and is unique

(vi) Subtraction of two complex numbers is defined as

$$z = z_1 + (-z_2)$$

1.2.2 Multiplication

Let $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$, then $z = z_1 \cdot z_2$ is defined as:

$$\begin{aligned} z &= (a_1 + ib_1)(a_2 + ib_2) \\ &= (a_1a_2 - b_1b_2) + i(a_1b_2 + a_2b_1) \end{aligned}$$

Properties of Complex Multiplication

(i) $z_1z_2 = z_2z_1$

(ii) $z_1(z_2z_3) = (z_1z_2)z_3$

(iii) $z_1(z_2 + z_3) = z_1z_2 + z_1z_3$

(iv) $1 \cdot z_1 = z_1$

(v) If $z_1 \neq 0$, then there exists a z such that $z_1 \cdot z = 1$

(vi) If $z_1 \neq 0$, then its inverse is unique

1.2.3 More Definitions

For $z = a + ib$, its complex conjugate is defined as:

$$\bar{z} = a - ib$$

and its "modulus" or "magnitude" is defined as:

$$\begin{aligned} |z| &= \sqrt{a^2 + b^2} \\ &= \sqrt{z \cdot \bar{z}} \\ &= \sqrt{z \cdot z^*} \end{aligned}$$

Properties of Complex Conjugation:

(i) $(z^*)^* = z$

(ii) $(z_1 + z_2)^* = z_1^* + z_2^*$

(iii) $(z_1z_2)^* = z_1^*z_2^*$

1.2.4 Triangle Equation

$\forall z_1, z_2 \in \mathbb{C}, |z_1 + z_2| \leq |z_1| + |z_2|$ Formally, $z_1 + z_2 = (a_1 + a_2) + i(b_1 + b_2)$, Thus,

$$\begin{aligned}|z_1 + z_2|^2 &= (a_1 + a_2)^2 + (b_1 + b_2)^2 \\ &= a_1^2 + a_2^2 + 2a_1a_2 + b_1^2 + b_2^2 + 2b_1b_2 \\ &\leq a_1^2 + a_2^2 + 2|a_1||a_2| + b_1^2 + b_2^2 + 2|b_1||b_2| \\ &= (a_1^2 + b_1^2) + (a_2^2 + b_2^2) + 2\sqrt{(a_1^2 + b_1^2)(a_2^2 + b_2^2)}\end{aligned}$$

Since, $|a_1||a_2| \leq \sqrt{(a_1^2 + b_1^2)(a_2^2 + b_2^2)}$

$$\begin{aligned}|z_1 + z_2|^2 &\leq a_1^2 + a_2^2 + 2|a_1||a_2| + b_1^2 + b_2^2 + 2|b_1||b_2| \\ &= (\sqrt{a_1^2 + b_1^2} + \sqrt{a_2^2 + b_2^2})^2 \\ &= (|z_1| + |z_2|)^2\end{aligned}$$

Thus, $|z_1 + z_2| \leq |z_1| + |z_2|$

1.3 Elementary Complex Functions

$$\begin{aligned}w(z) &= w(x + iy) \\ &= u(x, y) + iv(x, y)\end{aligned}$$

Chapter 2

Systems of Linear Algebraic Equations

Modelling physical systems in terms of linear systems of equation and obtaining solutions of these systems is of fundamental importance in engineering.

General form of a system of linear algebraic equation is:

$$\left.\begin{array}{c} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = c_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = c_2 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = c_n \end{array}\right\} \text{m equations}$$

a_{ij} : coefficients
c_i : values
x_j : unknowns

- If we replace x_1 with s_1, x_2 and $s_2, \ldots,$ x_n with s_2 in this system and satisfy all the equations, we say that $s_1, \ldots, s_n$ is a solution to the given system.
- If there exists one or more solutions, the system is *consistent*
- If there is precisely one solution, the solution is called as *unique*
- If there are more than one solutions, the solution is called as *non-unique*
- If there are no solutions, the system is called as *inconsistent*
- The collection of all solution is called as *the solution set*

In general, each equation defines a hyperplane in n-dimensional space. The system is consistent if these hyperplanes has a common intersection. If the intersection of these hyperplanes is just one point, then the solution is unique.

2.1 Gauss or Gaussian Elimination

Definition:

1. Addition of a multiple of one equation to another symbolically: $(eq_j) \to (eq_j) + \alpha(eq_k)$
2. Multiplication of an equation by a non-zero constant symbolically: $(eq_j) \to \alpha(eq_j)$
3. Interchange of two equations symbolically: $(eq_j) \leftrightarrow (eq_k)$

Theorem: If one linear system is obtained from another by a finite number of elementary row operations, then the two systems are equivalent, i.e., they share the same solution set.

Proof: Let the original system be LS_1 and the one obtained by using elementary row operations be LS_2. We know that $LS_1 \xrightarrow{E_1} \dots \xrightarrow{E_q} LS_2$, where $E_1, \dots, E_q$ is a sequence of elementary row operations. Here, by using the method of induction, we will show that LS_1 and LS_2 share the same solution set for $q = 1$:

$$LS_1 \xrightarrow{E_1} LS_1$$

Now, E_1 can be any of the 3 elementary row operation, we will consider all of these cases individually:

(i) E_1: addition of a multiple of one equation to another:

$$(eq_j) \to (eq_j) + \alpha(eq_q)$$

- Now, if $(s_1, \dots, s_n) \in S_{LS_1}$, then $(s_1, \dots, s_n)$ satisfies all the equations including (eq_j) and (eq_k). Hence, it satisfies the j^{th} equation of LS_2 which is $(eq_j) + \alpha(eq_k)$. Therefore:

$$S_{LS_1} \subset S_{LS_2}$$

- Now, if $(s_1, \dots, s_n) \in S_{LS_2}$, then $(s_1, \dots, s_n)$ satisfies all the equations including (eq_j) and (eq_k) of LS_2. Thus it also satisfies $(eq_j) - \alpha(eq_k)$ as well. But this is the (eq_j) of LS_1. Thus, $(s_1, \dots, s_n)$ also satisfies all the equations in LS_1. Hence $(s_1, \dots, s_n) \in S_{LS_1}$. Therefore

$$S_{LS_2} \subset S_{LS_1}$$

Since

$$S_{LS_2} \subset S_{LS_1} \text{ and } S_{LS_1} \subset S_{LS_2} \implies S_{LS_2} = S_{LS_1}$$

(ii) E_1: Multiplication of an equation by a non-zero constant

$$(eq_j) \to \alpha(eq_j)$$

- $(s_1, \dots, s_n) \in S_{LS_1}$ then it solves all the equations in LS_1. The LS_2 differs from LS_1 in just its j^{th} equation. Therefore, to show that $(s_1, \dots, s_n) \in S_{LS_2}$ also satisfies the (eq_j) of LS_2. Since (eq_j) in LS_2 is obtained by multiplying both sides of (eq_j) of LS_1, and $(s_1, \dots, s_n)$ satisfies this equation, $(s_1, \dots, s_n)$ also satisfies (eq_j) of LS_2. Therefore $(s_1, \dots, s_n) \in S_{LS_2}$ as well. Therefore $S_{LS_1} \subset S_{LS_2}$

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

Chapter 3

Linear Constant Coefficient Difference Equations

3.1 Introduction

$$\underbrace{x[n] = a_1 x[n-1] + \cdots + a_k x[n-k]}_{k^{th} \text{ order difference equation}}$$

Occurs very frequently discrete time systems or discretized models of continuous time systems. Can investigate the solution based on the following vector matrix relationship:

$$\underline{x}_n = \begin{bmatrix} x[n] \\ \vdots \\ x[n-k+1] \end{bmatrix}, \; \underline{\underline{A}} = \begin{bmatrix} a_1 & \cdots & \cdots & a_k \\ 1 & & & 0 \\ & \ddots & & \vdots \\ & & 1 & 0 \end{bmatrix} = \begin{bmatrix} [a_1 & \cdots & a_k] \\ \underline{\underline{I}} & & \underline{0} \end{bmatrix} \Longrightarrow \underline{x}_n = \underline{\underline{A}}\underline{x}_{n-1}$$

Since $\underline{x}_{n-1} = \underline{\underline{A}}\underline{x}_{n-2}$ we can write:

$$\underline{x}_n = \underline{\underline{A}}(\underline{\underline{A}} \ldots (\underline{\underline{A}} \cdot \underline{x}_0)) = \underline{\underline{A}}^x \underline{x}_0$$

where

$$\underline{x}_0 = \left.\begin{bmatrix} x[0] \\ \vdots \\ x[-k+1] \end{bmatrix}\right\} \text{initial conditions}$$

Therefore, given $\underline{x}_0$, can find $\underline{x}_n$ for any n by multiplying $\underline{x}_0$ n times wit $\underline{\underline{A}}$. But this is not the most efficient technique, and also would not provide us a closed form solution. We can get more insight as follows:

3.2 Eigenvalue Solution

Theorem: if $(\lambda, \underline{x}_0)$ is an eigenpair of $\underline{\underline{A}}$, then $\underline{x}_n = \lambda^n \underline{x}_0$ is a solution to $\underline{x}_n = \underline{\underline{A}}^n \underline{x}_0$

Proof: $\underline{x}_n = \underline{\underline{A}}^n \underline{x}_0 = \lambda^n \underline{x}_0$

Theorem: If $\underline{\underline{A}}$ has a full set of eigenvectors that span $\mathbb{R}^k$, then $\underline{x}_n = \underline{\underline{A}}^n \underline{x}_0$ can be solved by

$$\underline{x}_n = \sum_{i=1}^{k} = \alpha_i \lambda_i^n \underline{v}_i \text{ where } \underline{x}_0 = \sum_{i=1}^{k} \alpha_i \underline{v}_i$$

Proof: Simply multiply $\underline{x}_0$ by $\underline{\underline{A}}^n$ to get

$$\underline{\underline{A}}^n \underline{x}_0 = \sum_{i=1}^{k} \alpha_i \underline{\underline{A}}^n \underline{v}_i = \sum_{i=1}^{k} \alpha_i \lambda_i^n \underline{v}_i$$

3.3 Characteristic Equation Solution

Another alternative approach is based on the following observation:

$$x[n] = a_1 x[n-1] + a_2 x[n-2] + \cdots + a_k x[n-k]$$

A sequence of the form $x[n] = r^n$ would solve the difference equation if:

$$r^n = a_r^{n-1} + \cdots + a_k r^{n-k} \text{ or } r^n -_r^{n-1} - \cdots - a_k r^{n-k} = 0, \ \forall n$$

$$\implies r^{n-k} \underbrace{(r^k - a_1 r^{k-1} - \cdots - a_k)}_{\text{characteristic equation = 0 at r}} = 0$$

$$\implies \text{r should be a root of the characteristic equation}$$

This is equivalent to r being and eigenvalue of the $\underline{\underline{A}}$. Hence, there can be at most k distinct r values. If they are all distinct, then a solution to the original problem be formulated easily by:

$$x[n] = \alpha_1 r_1^n + \cdots + \alpha_k r_k^n = \sum_{i=1}^{k} \alpha_i r_i^k, \ n \geq 0$$

where if we are given a set of initial conditions such as

$$\begin{bmatrix} x[0] \\ \vdots \\ x[-k+1] \end{bmatrix} \text{ or } \begin{bmatrix} x[-1] \\ \vdots \\ x[-k] \end{bmatrix}$$

we can find α_i's as the solution to

$$\underbrace{\begin{bmatrix} r_1^{-1} & \dots & r_k^{-1} \\ \vdots & \ddots & \vdots \\ r_1^{-k} & \dots & r_k^{-1} \end{bmatrix}}_{\text{invertible}} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_k \end{bmatrix} = \begin{bmatrix} x[-1] \\ \vdots \\ x-k \end{bmatrix}$$

If a root r_j has the multiplicity m, then solution can be written as

$$x[n] = \alpha_1 r_1^n + \cdots + \alpha_j r_j^n + \alpha_{j+1} n r_j^n + \cdots + \alpha_{j+m} n^{m-1} r_j^n + \cdots + \alpha_{m+k} r_k^n$$

3.4 Z-Transform

A very useful transformation that is commonly used to investigate and design of discrete time systems. It is also useful in the solution of difference equations.

Definition: Given a sequence $x[n]$, it is also $\mathbb{X}(z) is defined as$

$$\mathbb{X}(z) = \sum_{n=0}^{\infty} x[n] z^{-n}$$

when the summation converges.

Facts

1. For finite length sequences z transform converges for any z, except may be $z = 0$.
2. If the z-transform of a sequence converges, it converges for al $|z| > r_0$, where r_0 depends on the sequence.

Definition: Those z-values for which $\mathbb{X}(F)$ is defined are said to be in the "Region of Convergence", or ROC.

Properties

(i) z-transform is linear: $\forall x_1[n],\ x_2[n]\ \alpha_1\ and\ \alpha_2$

$$x_1[n] \overset{Z}{\longleftrightarrow} \mathbb{X}_1(z),\ ROC_1$$
$$x_2[n] \overset{Z}{\longleftrightarrow} \mathbb{X}_2(z),\ ROC_2$$

Then

$$\alpha_1 x_1[n] + \alpha_2 x_2[n] \overset{Z}{\longleftrightarrow} \mathbb{X}_1(z) + \mathbb{X}_2(z),\ ROC_1, ROC_2 \subset ROC$$

(ii)

$$x[n] \overset{Z}{\longleftrightarrow} \mathbb{X}(z)$$
$$x[n-1] \overset{Z}{\longleftrightarrow} z^{-1}\mathbb{X}(z) + x[-1]$$

(iii)

$$x[n] \overset{Z}{\longleftrightarrow} \mathbb{X}(z)$$
$$x[n-k] \overset{Z}{\longleftrightarrow} z^{-k}\mathbb{X}(z) + z^{-k+1}x[-1] + z^{-k+2}x[-2] + \cdots + x[-k]$$

(iv)

$$\begin{aligned} x[n] &= a^n,\ n \geq 0 \\ \mathbb{X}(z) &= \sum_{n=0}^{\infty} a^n z^{-n} \\ &= \sum_{n=0}^{\infty} (az^{-1})^n \\ &= \frac{1}{1-az^{-1}},\ |az^{-1}| < 1 \implies |a| < |z| :\ ROC \end{aligned}$$

(v)

$$\begin{aligned} x[n] &= na^n,\ n \geq 0 \\ \mathbb{X}(z) &= \sum_{n=0}^{\infty} na^n z^{-n} \\ &= \left[\sum_{n=0}^{\infty} na^n z^{-(n+1)}\right] \cdot z \\ &= \left(-\frac{d}{dz}\sum_{n=0}^{\infty} a^n z^{-n}\right) \cdot z \\ &= -\left[\frac{d}{dz}\frac{1}{1-az^{-1}}\right] \cdot z \\ \mathbb{X}(z) &= \frac{az^{-1}}{(1-az^{-1})^2} \end{aligned}$$

Note: If the forcing input is one of the eigenvalues, it causes a resonance in the solution, then the z-transform of it will be multiplied with n.

Chapter 4

Linear Constant Coefficient Differential Equations

4.1 Homogeneous Differential Equation

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1}\frac{dy}{dx} + a_n y = 0 \Big\{ \text{Homogenous } n^{th} \text{ order Linear, Constant Coeff. DE}$$

4.1.1 First Order Case

$$\frac{dy}{dx} + a_1 y = 0$$

Consider a solution in the form $y(x) = e^{\lambda x}$, then

$$\frac{dy}{dx} = \lambda e^{\lambda x} \implies \lambda e^{\lambda x} + a_1 e^{\lambda x} = 0 \implies a_1 = -\lambda$$

Hence, the general solution is of the form $Ce^{-a_1 x}$. To determine C, we need an initial condition such as $y(0)$. Then $y(0) = C \implies y(x) = y(0)e^{-a_1 x}$

4.1.2 Second Order Case

$$\frac{d^2 y}{dx^2} + a_1 \frac{dy}{dx} + a_2 y = 0$$

Consider a solution in the form $y(x) = e^{\lambda x}$

$$\lambda^w e^{\lambda x} + a_1 \lambda e^{\lambda x} + a_2 e^{\lambda x} = 0 \implies (\lambda^2 + a_1\lambda + a_2)e^{\lambda x} = 0 \implies \underbrace{\lambda^2 + a_1\lambda + a_2}_{\text{characteristic equation}}$$

$$\lambda_1 = \frac{-a_1 + \sqrt{a_1^2 - 4a_2}}{2},\ \lambda_2 = \frac{-a_1 - \sqrt{a_1^2 - 4a_2}}{2}$$

for real a_1 and a_2, we may have two real roots or a pair of complex conjugate roots the general solution is of the form

$$y(x) = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}$$

If we are given initial conditions such as $y(0)$ and $y'(0)$ can uniquely determine C_1 and C_2.

Complex Roots Case

For complex root in the form $\lambda_1 = a + jb$ and it's conjugate $\lambda_2 = a - jb$, we add the equation

$$C_1 e^{a+jb} x + C_2 e^{a-jb} x = e^{ax}(A\cos(bx) + B\sin(bx))$$

to the solution of the differential equation.

Theorem: If the characteristic equation of an n^{th} order linear constant coefficient differential equation has n-distinct roots, $\lambda_1, \dots, \lambda_n$ the set $\{e^{\lambda_1 x}, \dots, e^{\lambda_n x}\}$ is linearly independent in any interval $\mathbb{I} \subset \mathbb{R}$.

Proof: Assume that they are linearly dependent. Then, there should be a set of coefficients not all zero such that

$$\alpha_1 e^{\lambda_1 x} + \alpha_2 e^{\lambda_2 x} + \cdots + \alpha_n e^{\lambda_n x} = 0, \ \forall x \in \mathbb{I}$$

By taking successive derivatives of both sides we get:

$$\alpha_1 \lambda_1 e^{\lambda_1 x} + \alpha_2 \lambda_2 e^{\lambda_2 x} + \cdots + \alpha_n \lambda_n e^{\lambda_n x} = 0$$

$$\vdots$$

$$\alpha_1 \lambda_1^{n-1} e^{\lambda_1 x} + \alpha_2 \lambda_2^{n-1} e^{\lambda_2 x} + \cdots + \alpha_n \lambda_n^{n-1} e^{\lambda_n x} = 0$$

$$\implies \underbrace{\begin{bmatrix} 1 & 1 & \dots & 1 \\ \lambda_1 & \lambda_2 & \dots & \lambda_n \\ \vdots & \vdots & & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \dots & \lambda_n^{n-1} \end{bmatrix}}_{\text{Full Rank Vandermonde Matrix for } \lambda_i \neq \lambda_j} \cdot \begin{bmatrix} \alpha_1 e^{\lambda_1 x} \\ \alpha_2 e^{\lambda_1 x} \\ \dots \\ \alpha_n e^{\lambda_n x} \end{bmatrix} = \underline{0} \implies \begin{bmatrix} \alpha_1 e^{\lambda_1 x} \\ \alpha_2 e^{\lambda_1 x} \\ \dots \\ \alpha_n e^{\lambda_n x} \end{bmatrix} = \underline{0} \implies \alpha_i e^{\lambda_i x} = 0 \implies \alpha_i = 0$$

which is a contradiction since all of the coefficients are zero.

Repeated Roots Case

Theorem: If λ_1 is a root of order k of the characteristic equation, then $e^{\lambda_1 x}, xe^{\lambda_1 x}, \dots, x^{k-1}e^{\lambda_1 x}$ are linearly independent solutions of the differential equation.

Proof: Let $L[y] = 0$ be the operator form of

$$L[b] = \frac{d^n b}{dx^n} + a_1 \frac{d^{n-1} b}{dx^{n-1}} + \cdots + a_{n-1}\frac{db}{dx} + a_n b$$

Then $L[e^{\lambda x}] = (\lambda - \lambda_1)^k p(\lambda) e^{\lambda x}$. Since λ_1 is a root with multiplicity k and order of $p(\lambda)$ is $(n-k)$.

$$\implies L[e^{\lambda_1 x}] = 0 \implies e^{\lambda_1 x} \text{ is a solution}$$

Now, want to show that $xe^{\lambda_1 x}$ is also a solution

$$\frac{d}{d\lambda} L[e^{\lambda x}] = \frac{d}{d\lambda}[(\lambda - \lambda_1)^k p(\lambda)e^{\lambda x}] = k(\lambda - \lambda_1)^{k-1} p(\lambda) e^{\lambda x} + (\lambda - \lambda_1)^k \frac{d}{d\lambda}(p(\lambda)e^{\lambda x})$$

Note that:

$$\frac{d}{d\lambda} L[e^{\lambda x}] = L\left[\frac{d}{d\lambda} e^{\lambda x}\right] = L[xe^{\lambda x}]$$

$$\implies L[xe^{\lambda x}]\big|_{\lambda=\lambda_1} = \underbrace{k(\lambda - \lambda_1)^{k-1} p(\lambda) e^{\lambda x} + (\lambda - \lambda_1)^k \frac{d}{d\lambda}(p(\lambda)e^{\lambda x})\Big|_{\lambda=\lambda_1}}_{0}$$

$$\implies L[xe^{\lambda_1 x}] = 0 \implies xe^{\lambda_1 x} \text{ is also a solution.}$$

Can apply this same procedure to prove that $x^2 e^{\lambda_1 x}, \dots, x^{k-1} e^{\lambda_1 x}$ are also solution.

Chapter 5

Laplace Transforms

5.1 Introduction

General linear integral transform is of the following form

$$F(s) = \int_a^b K(t,s)f(t)dt,\ K(t,s):\ \text{Transformation kernel}$$

The Laplace Transform:

$$\boxed{F(s) = \int_0^\infty f(t)e^{-st}dt}$$

s : complex valued transform domain variable.

Similar to z-transform, which reduces difference equations to linear algebraic equations, the Laplace Transform reduces the linear constant coefficient differential equations to linear algebraic equations.

5.2 Calculation of the Transform

$$F(s) = \int_0^\infty e^{-st}dt$$

convergence is assured if $|f(t)| \leq Ke^{ct}$ for $t > T$

*Theorem:*For $f(t)$ satisfying

(i) $f(t)$ is piecewise continuous on $0 \leq t \leq A$, with finite number of discontinuities.

(ii) $f(t)$ is of exponential order, $|f(t)| \leq Ke^{ct}$, $t \geq T$, the Laplace Transform $F(s)$ exists for $\text{Re}\{s\} > c$.

Proof:

$$F(s) = \int_0^\infty f(t)e^{-st}dt = \underbrace{\int_0^T f(t)e^{-st}dt}_{\text{exists for all s if (i) is valid}} + \int_T^\infty f(t)e^{-st}dt$$

$$\left|\int_T^\infty f(t)e^{-st}dt\right| \leq \int_T^\infty |f(t)||e^{-st}|dt \leq K\int_t^\infty e^{-(\text{Re}\{s\}-c)t}dt = K\frac{e^{-(\text{Re}\{s\}-c)T}}{\text{Re}\{s\}-c}$$

for $\text{Re}\{s\} > c$

<u>**Notes**</u>:

(i) The inverse Laplace Transform is also an integral transform but requires techniques that will be introduced in EEE-242. Therefore, we will use the inspection technique like we did with the inversion of the z-transform.

(ii) The inverse of the Laplace Transform is unique if

$$f_1(t) \overset{L}{\longleftrightarrow} F_1(s)$$
$$f_2(t) \overset{L}{\longleftrightarrow} F_2(s)$$

and

$$\int_0^\infty |f_1(t) - f_2(t)| dt > 0$$

then $F_1(s) \neq F_2(s)$

5.3 Properties of Laplace Transform

(i) $L\{\alpha u(t) + \beta v(t)\} = \alpha L\{u(t)\} + \beta L\{v(t)\}$ is satisfied for $\forall \alpha, \beta, u(t)$ and $v(t)$ which are of exponential order.

(ii) $L^{-1}\{\alpha U(s) + \beta V(s)\} = \alpha L^{-1}\{U(s)\} + \beta L^{-1}\{V(s)\}$

(iii) $L\{f'(t)\} = sL\{f(t)\} - f(0)$

Proof:

$$\begin{aligned} L\{f'(t)\} &= \int_0^\infty f'(t) e^{-st} dt \\ &= f(t)e^{-st}\Big|_0^\infty + s\int_0^\infty f(t)e^{-st}dt \\ &= -f(0) + sF(s) \end{aligned}$$

Note that this property is valid even if $f'(t)$ is piecewise continuous.

(iv)

$$\begin{aligned} L\{f''(t)\} &= sL\{f'(t)\} - f'(0) \\ &= s^2 L\{f(t)\} - sf(0) - f'(0) \\ \implies L\{f^{(k)}(t)\} &= s^k L\{f(t)\} - \sum_{i=1}^{k} s^{k-i} f^{(i-1)}(0) \end{aligned}$$

useful in the solution to differential equation with given initial condition.

(v) *Translation:* $L\{e^{at} f(t)\} \overset{L}{\longleftrightarrow} F(s-a)$

(vi) *Translation in Time:* $L\{f(t-t_0)\} \overset{L}{\longleftrightarrow} e^{-st_0}F(s)$

Proof:

$$\begin{aligned}
L\{f(t-t_0)\} &= \int_0^\infty f(\underbrace{t-t_0}_{\hat{t}})e^{-s(\hat{t}+t_0)}dt \\
&= \int_{t_0}^\infty f(\hat{t})e^{-s(\hat{t}+t_0)}d\hat{t} \\
&= e^{-s_0 t}F(s),\ f(\hat{t}) = 0 \text{ for } \hat{t} < 0
\end{aligned}$$

(vii)

$$L\{t^n f(t)\} = (-1)^n F^{(n)}(s)$$

(viii) *Convolution property:* For $f(t)$ and $g(t)$ which are zero for $t < 0$, their convolution is defined as:

$$h(t) = f(t) \circledast g(t) = (f \circledast g)(t) = \int_0^\infty f(\tau)g(t-\tau)d\tau$$

The Laplace Transform of $h(t)$ is:

$$\begin{aligned}
H(s) &= \int_0^\infty h(t)e^{-st} \\
&= \int_0^\infty \int_0^t f(\tau)g(t-\tau)d\tau\ e^{-st}dt \\
&= \int_0^\infty \int_\tau^\infty f(\tau)g(t-\tau)e^{-st}dtd\tau \\
&= \int_0^\infty f(\tau)\left[\int_0^\infty g(\underbrace{t-\tau}_{\mu})e^{-st}dt\right]d\tau \\
&= \int_0^\infty f(\tau)\left[\int_0^\infty g(\mu)e^{-s(\tau+\mu)}d\mu\right]d\tau \\
&= \int_0^\infty f(\tau)\left[\int_0^\infty g(\mu)e^{-s\mu}d\mu\right]e^{-s\tau}d\tau \\
&= \left[\int_0^\infty f(\tau)e^{-s\tau}d\tau\right]\left[\int_0^\infty g(\mu)e^{-s\mu}d\mu\right] \\
&= F(s)\cdot G(s) \implies \boxed{H(s) = F(s)\cdot G(s)}
\end{aligned}$$

5.3.1 Transformation Table

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$	$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1	$\frac{1}{s}$	e^{at}	$\frac{1}{s-a}$
t^n	$\frac{n!}{s^{n+1}}$	$\sqrt{t}$	$\frac{\sqrt{\pi}}{2s^{3/2}}$
$\sin(at)$	$\frac{a}{s^2+a^2}$	$\cos(at)$	$\frac{s}{s^2+a^2}$
$t\sin(at)$	$\frac{2as}{(s^2+a^2)^2}$	$t\cos(at)$	$\frac{s^2-a^2}{(s^2+a^2)^2}$
$\sin(at+b)$	$\frac{s\sin(b)+a\cos(b)}{s^2+a^2}$	$\cos(at+b)$	$\frac{s\cos(b)-a\sin(b)}{s^2+a^2}$
$\sinh(at)$			

5.4 Solution to Linear Constant Coefficient Differential Equations by Laplace Transform

Example

$$
\begin{aligned}
x'' + ax' + bx &= f(t) \\
&\updownarrow L \\
(s^2\mathbb{X}(s) - sx(0) - x'(0)) + a(s\mathbb{X}(s) - x(0)) + b\mathbb{X}(s) &= F(s) \\
\implies (s^2 + as + b)\mathbb{X}(s) &= sx(0) + ax(0) + x'(0) + F(s) \\
\implies \mathbb{X}(s) &= \underbrace{\frac{(s+a)x(0) + x'(0)}{s^2 + as + b}}_{\text{invert by partial fraction expansion}} + \underbrace{\frac{F(s)}{s^2 + as + b}}_{\text{invert by using partial fraction expansion or convolution property}}
\end{aligned}
$$

5.5 Systems of Linear Differential Equations

$$
\begin{aligned}
a_{11}(t)x_1' + \cdots + a_{1n}(t)x_n' + b_{11}(t)x_1 + \cdots + b_{1n}(t)x_n &= f_1(t) \\
&\vdots \\
a_{n1}(t)x_1' + \cdots + a_{nn}(t)x_n' + b_{n1}(t)x_1 + \cdots + b_{nn}(t)x_n &= f_n(t)
\end{aligned}
$$

$a_{ij}(t)$ and $b_{ij}(t)$ are known coefficients.

Theorem: Let $a_{ij}(t)$, $1 \le i, j \le n$ and $f_i(t)$, $1 \le i \le n$ be continuous on a closed interval $\mathbb{I}$. Also, let

$x_i(t_0) = b_i$, $1 \le i \le n$ for $t_0 \in \mathbb{I}$. Then the system

$$x_1' = a_{11}(t)x_1 + \ldots + a_{1n}(t)x_n + f_1(t)$$
$$\vdots$$
$$x_n' = a_{n1}(t)x_1 + \ldots + a_{nn}(t)x_n + f_n(t)$$

has a unique solution on the entire interval $\mathbb{I}$. Note that

(i) The left hand side has individual derivatives. This form can be obtained by using Gauss-Jordan elimination on the original system.

(ii) The system can be written as:

$$\underline{x}' = \underline{\underline{A}}(t)\underline{x} + \underline{f}$$

where

$$\underline{x} = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix},\ \underline{\underline{A}}(t) = [a_{ij}(t)]_{n\times n},\ \underline{f} = \begin{bmatrix} f_1(t) \\ \vdots \\ f_n(t) \end{bmatrix}$$